CONTACT HYPERSURFACES IN UNIRULED SYMPLECTIC MANIFOLDS ALWAYS SEPARATE

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ABSTRACT. We observe that nonzero Gromov-Witten invariants with marked point constraints in a closed symplectic manifold imply restrictions on the homology classes that can be represented by contact hypersurfaces. As a special case, contact hypersurfaces must always separate if the symplectic manifold is uniruled. This removes a superfluous assumption in a result of G. Lu [Lu00], thus implying that all contact manifolds that embed as contact type hypersurfaces into uniruled symplectic manifolds satisfy the Weinstein conjecture. We prove the main result using the Cieliebak-Mohnke approach to defining Gromov-Witten invariants via Donaldson hypersurfaces, thus no semipositivity or virtual moduli cycles are required.

CONTENTS

1. THE STATEMENT

1.1. **Main result and consequences.** In this note, we prove the following.

Main theorem. *Suppose (M, ω) is a closed symplectic manifold and $V \subset M$ is a real hypersurface that is pseudoconvex for some choice of ω-compatible almost complex structure on M. Then the rational Gromov-Witten invariants of (M, ω), defined in the sense of* [CM07] *(see §2.1.1 and §2.1.2), satisfy*

$$\mathrm{GW}_{0,m,A}^{(M,\omega)}(\mathrm{PD}[V] \cup \alpha_1, \alpha_2, \ldots, \alpha_m; \beta) = 0$$

for all $m \geq 3$, $A \in H_2(M)$, $\alpha_1, \ldots, \alpha_m \in H^*(M; \mathbb{Q})$ and $\beta \in H_*(\overline{\mathcal{M}}_{0,m}; \mathbb{Q})$.

Recall that a real hypersurface V in an almost complex manifold (M, J) is **pseudoconvex** (also sometimes called J-**convex**) if the maximal J-invariant subbundle $\xi \subset TV$ is a contact structure whose canonical conformal class of symplectic structures tames $J|_\xi$. As an important special case, when (M, ω) is a symplectic manifold, we say $V \subset M$ is a **contact type hypersurface** if ω can be written in a neighborhood of V as $d\lambda$ for some 1-form λ whose restriction to V is a contact form. In that case, V is J-convex for any choice of ω-tame almost complex structure J that preserves the contact structure on V, and without loss of generality one can also arrange J to be ω-compatible.

We will show in §1.2 below that the main theorem has the following immediate consequence:

Corollary 1.1. *Suppose (M, ω) is a closed symplectic manifold that is symplectically uniruled (see Definition 1.5). Then every contact type hypersurface in (M, ω) is separating.*

Some motivation to prove such a result comes from the Weinstein conjecture, which asserts that any closed contact type hypersurface in a symplectic manifold has a closed orbit of its characteristic line field. There is a long history of results that prove this conjecture under various assumptions on the existence of holomorphic curves in the ambient symplectic manifold, cf. [HV92, LT00, Lu00]. However, such results have often been proved only for *separating* contact hypersurfaces, leaving the question without this extra assumption open. Our theorem thus shows that the extra assumption is superfluous, e.g. combining it with Guangcun Lu's result, we obtain:

Corollary 1.2 (via [Lu00])**.** *If (V, ξ) is a contact manifold that embeds into a symplectically uniruled symplectic manifold as a contact type hypersurface, then every contact form for (V, ξ) admits a periodic Reeb orbit, i.e. the Weinstein conjecture holds for (V, ξ).*

For more on symplectic manifolds to which this result applies, see [Hyv12] and the references therein.

Remark 1.3. Our use of the technique of Cieliebak and Mohnke [CM07] for defining the Gromov-Witten invariants via Donaldson hypersurfaces imposes certain technical restrictions on the scope of the above results: (1) The setup in [CM07] only handles symplectic manifolds with integral cohomology, i.e. $[\omega] \in H^2(M; \mathbb{Z})$, due to the need for a symplectic hypersurface Poincaré dual to a large multiple of $[\omega]$. One can obviously generalize this to the assumption that $[\omega]$ is any real multiple of an integral class, and of course every symplectic form admits a small perturbation that has this property. It is likely moreover that the restriction to integral classes can be lifted entirely by choosing symplectic hypersurfaces that approximate the relevant homology classes, and indeed, the recent preprint of Ionel and Parker [IP] claims to define fully deformation-invariant Gromov-Witten invariants for arbitrary $[\omega] \in H^2_{\mathrm{dR}}(M)$ using similar techniques. For simplicity, we shall nonetheless assume wherever necessary that $[\omega]$ is integral, in order to remain fully consistent with [CM07]. (2) Following [MNW13], one can define a real hypersurface V in a symplectic manifold (M, ω) to be **weakly contact** if there exists an ω-tame almost complex structure J for which V is

J-convex. This is equivalent to the condition required in our main theorem if $\dim V = 3$, but in higher dimensions it appears to be more general. It is very likely that our main theorem holds under this weaker assumption as well, and the proof given here will imply this at least in the semipositive case without coupling to gravity (using the standard setup from [MS04]). A more general proof will probably be possible in the future using polyfolds (cf. Remark 1.6). In the non-semipositive case, our reliance on the Donaldson hypersurface construction [Don96] necessitates the added restriction that J is *compatible* with ω, not just tamed.

1.2. **Recollections on Gromov-Witten theory.** In this article, we regard the *Gromov-Witten invariants* of a symplectic manifold (M, ω) as an association to each pair of integers $g, m \geq 0$ with $2g + m \geq 3$ and each homology class $A \in H_2(M)$ of a homomorphism

$$(1.1) \qquad \mathrm{GW}_{g,m,A}^{(M,\omega)} : H^*(M; \mathbb{Q})^{\otimes m} \otimes H_*(\overline{\mathcal{M}}_{g,m}; \mathbb{Q}) \to \mathbb{Q},$$

where $\overline{\mathcal{M}}_{g,m}$ denotes the Deligne-Mumford compactification of the moduli space of Riemann surfaces with genus g and m marked points. Let

$$\mathrm{PD} : H_*(M; \mathbb{Q}) \to H^*(M; \mathbb{Q})$$

denote the Poincaré duality isomorphism, or its inverse when convenient. In the absence of transversality problems, $\mathrm{GW}_{g,m,A}^{(M,\omega)}(\alpha_1, \ldots, \alpha_m; \beta)$ is interpreted as a count of rigid unparametrized J-holomorphic curves of genus g, for a generic ω-tame almost complex structure J, with m marked points such that for $i = 1, \ldots, m$, the ith marked point is mapped to a generic smooth representative of $\mathrm{PD}(\alpha_i) \in H_*(M)$, and the underlying conformal structure of the domain lies in a generic smooth representative of $\beta \in H_*(\overline{\mathcal{M}}_{g,m})$. In practice, the transversality problems that arise in this definition require considerable effort to overcome, and the literature contains various approaches (e.g. [FO99, LT98, Rua99, Sie, CM07, HWZ]) which may or may not all define the same invariants.

In order to be concrete and also minimize the technical apparatus needed, in this paper we shall work with the definition provided by Cieliebak and Mohnke [CM07] for the $g = 0$ case, which uses a *Donaldson hypersurface* as auxiliary data and thus requires the symplectic form to represent an integral cohomology class. The essential details of this setup will be reviewed in §2.1.2, though we shall also attempt to express the main argument in terms that do not depend on these details. In particular, the reader who would prefer to avoid serious technical issues by assuming (M, ω) is semipositive may do so by skipping from §2.1.1 (where we review the main definitions in the semipositive case) straight to §3. In either case, the theory is defined essentially by constructing a suitably compactified moduli space $\overline{\mathcal{M}}_{0,m}^A(M, J)$ of stable nodal pseudoholomorphic spheres homologous to A, with m marked points, such that the natural *evaluation/forgetful map*

$$(1.2) \qquad (\mathrm{ev}, \Phi) = (\mathrm{ev}_1, \ldots, \mathrm{ev}_m, \Phi) : \mathcal{M}_{0,m}^A(M, J) \to M^m \times \overline{\mathcal{M}}_{0,m}$$

defines a *rational pseudocycle* in the sense of [MS04, §6.5], meaning that rational intersection numbers with homology classes in $M^m \times \overline{\mathcal{M}}_{0,m}$ can be

defined. The homomorphism (1.1) is then defined, up to a combinatorial constant (see (2.4)), by

$$(1.3) \quad \mathrm{GW}^{(M,\omega)}_{0,m,A}(\alpha_1,\ldots,\alpha_m;\beta) = [(\mathrm{ev},\Phi)] \cdot (\mathrm{PD}(\alpha_1) \times \ldots \times \mathrm{PD}(\alpha_m) \times \beta) \,.$$

Remark 1.4. The Gromov-Witten invariants defined in [CM07] do not involve "coupling to gravity," i.e. they rely on the fact that $\mathrm{ev} : \mathcal{M}^A_{0,m}(M,J) \to M^m$ is a pseudocycle, but do not deal at all with the forgetful map $\Phi : \mathcal{M}^A_{0,m}(M,J) \to \overline{\mathcal{M}}_{0,m}$, associating to a J-holomorphic curve its underlying conformal structure. It is nonetheless true in the context of [CM07] that (ev,Φ) is a pseudocycle and hence (1.3) is well defined; the proof of this fact is almost already implicit in that paper, and we shall spell out the missing ingredients in Appendix A. Note that in the semipositive case, the standard approach via domain-dependent almost complex structures suffices to prove that the evaluation map is a pseudocycle, but not the forgetful map—see [MS04, pp. 184–186]. Thus the simplified version of our arguments (avoiding Donaldson hypersurfaces) for the semipositive case will be valid only for the simplified invariants $\mathrm{GW}^{(M,\omega)}_{0,m,A} : H^*(M;\mathbb{Q})^{\otimes m} \to \mathbb{Z}$, which match (1.1) if β is defined as the fundamental class of $\overline{\mathcal{M}}_{0,m}$.

We now recall the following standard definition.

Definition 1.5. A closed symplectic manifold (M,ω) is said to be **symplectically uniruled** if it has a nonzero rational Gromov-Witten invariant with at least one pointwise constraint, i.e. there exist $A \in H_2(M)$, an integer $m \geq 3$ and classes $\alpha_2,\ldots,\alpha_m \in H^*(M;\mathbb{Q})$, $\beta \in H_*(\overline{\mathcal{M}}_{0,m};\mathbb{Q})$ such that

$$(1.4) \qquad\qquad \mathrm{GW}^{(M,\omega)}_{0,m,A}(\mathrm{PD}[\mathrm{pt}],\alpha_2,\ldots,\alpha_m;\beta) \neq 0,$$

where $[\mathrm{pt}] \in H_0(M)$ denotes the homology class of a point.

Morally, being symplectically uniruled means one can find a set of constraints so that there is always a nonzero count of constrained holomorphic spheres passing through a generic point.

Proof of Corollary 1.1. If $V \subset M$ is a nonseparating hypersurface, then $[V] \neq 0 \in H_*(M;\mathbb{Q})$ and one can therefore find a cohomology class $\alpha_1 \in H^*(M;\mathbb{Q})$ with $\langle \alpha_1, [V] \rangle = 1$. Hence

$$\mathrm{PD}[V] \cup \alpha_1 = \mathrm{PD}[\mathrm{pt}].$$

Now if V is also pseudoconvex for some compatible almost complex structure, then the main theorem implies that (1.4) cannot be satisfied for any choices $\alpha_2,\ldots,\alpha_m,\beta$, hence (M,ω) is not uniruled. $\qquad\square$

Remark 1.6. An earlier version of the present paper made the optimistic claim that the arguments given here can be carried out using the polyfold theory of Hofer-Wysocki-Zehnder [HWZ]. While that is probably true, subsequent discussions with Hofer have led to the conclusion that it is not fully provable using the technology in its present state: in particular, homological intersection theory and Poincaré duality are not currently well enough understood in the polyfold context to justify anything analogous to Equation (3.2). I would like to thank Joel Fish and Helmut Hofer for helping clarify this point.

1.3. **Discussion.** We now add a few more remarks on the context of the main theorem and its corollaries.

1.3.1. *Nonseparating hypersurfaces.* Nonseparating contact type hypersurfaces do exist in general, though they are usually not easy to find. A construction in dimension 4 was suggested by Etnyre and outlined in [ABW10, Example 1.3]: the idea is to start from a symplectic filling with two boundary components, attach a Weinstein 1-handle to form the boundary connected sum and then attach a symplectic cap to form a closed symplectic manifold, which contains both boundary components of the original symplectic filling as nonseparating contact hypersurfaces. At the time [ABW10] was written, examples of symplectic fillings with disconnected boundary were known only up to dimension 6 (due to McDuff [McD91], Geiges [Gei95, Gei94] and Mitsumatsu [Mit95]), but recently a construction in all dimensions appeared in work of the author with Massot and Niederkrüger [MNW13]. It seems likely that these examples can be combined with the symplectic capping result of Lisca and Matić [LM97, Theorem 3.2] for Stein fillable contact manifolds to construct examples of nonseparating contact hypersurfaces in all dimensions, but we will not pursue this any further here.

Note that it is somewhat easier to find examples of *weakly* contact hypersurfaces that do not separate: for instance, considering the standard symplectic $\mathbb{T}^4$ as a product of two symplectic 2-tori, for any nonseparating loop $\gamma \subset \mathbb{T}^2$ the hypersurface $\gamma \times \mathbb{T}^2 \subset \mathbb{T}^4$ admits an obvious foliation by symplectic 2-tori, and this foliation can be perturbed to any of the tight contact structures on $\mathbb{T}^3$ (cf. [Gir94]). Notice that one cannot use the same trick to produce a nonseparating weakly contact hypersurface in $\mathbb{T}^2 \times S^2$ with any product symplectic structure, as the latter is uniruled.[1] This implies the well known fact (see [ET98]) that the obvious foliation by spheres on $S^1 \times S^2$ cannot be perturbed to a contact structure.

1.3.2. *Higher genus.* The theorem of Lu [Lu00] also establishes the Weinstein conjecture for separating contact type hypersurfaces under the more general assumption

$$(1.5) \qquad \mathrm{GW}_{g,m,A}^{(M,\omega)}(\mathrm{PD}([\mathrm{pt}]), \alpha_2, \ldots, \alpha_m; \beta) \neq 0,$$

i.e. one need not assume $g = 0$. In fact, using the more recent technology of "stretching the neck" [BEH+03], one can give a straightforward alternative proof of Lu's result which also shows that any *nonseparating* contact hypersurface in a manifold satisfying (1.5) must have a closed characteristic.[2] Note however that in the genus zero case, this is a weaker statement than Corollary 1.2: it asserts that a *particular* contact form on $(V, \xi) \subset (M, \omega)$ admits a closed Reeb orbit, but not that this is true for every possible choice of contact form. The obvious stretching argument does not appear to imply this stronger statement in general except when V separates M.

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[1] Actually, the statement of our main theorem for $\mathbb{T}^2 \times S^2$ can be proved by more elementary means without mentioning Gromov-Witten invariants, cf. [ABW10, Theorem 1.15].

[2] For this heuristic discussion we are ignoring the usual analytical issues of how to define the higher genus Gromov-Witten invariants; definitions using the Donaldson hypersurface idea have appeared in recent work of Gerstenberger [Ger13] and Ionel-Parker [IP].

It seems unlikely moreover that our main result would hold under the more general assumption (1.5)—certainly the method of proof given below does not work, as it requires the fact that the relevant holomorphic curves in M can always be lifted to a cover (since S^2 is simply connected). However, it was pointed out to me by Guangcun Lu that due to relations among Gromov-Witten invariants (see [Lu06, §7]), certain conditions on higher genus invariants will imply that (M, ω) is also uniruled, e.g. this is the case whenever there is a nontrivial invariant of the form

$$\text{GW}^{(M,\omega)}_{g,m,A}(\text{PD}([\text{pt}]), \alpha_2, \ldots, \alpha_m; [\text{pt}]) \neq 0.$$

The reason is that this invariant counts curves with a fixed conformal structure on the domain, so one can derive holomorphic spheres from them by degenerating the conformal structure to "pinch away" the genus.

Remark 1.7. Note that in the above formulation of the Weinstein conjecture for closed contact hypersurfaces, the ambient symplectic manifold need not be closed, e.g. every contact manifold is a contact hypersurface in its own (noncompact) symplectization. As was shown in [ABW10], there are many contact manifolds that do not admit any contact type embeddings into any closed symplectic manifold—as far as I am aware, all contact manifolds that are currently known to admit such embeddings are also symplectically fillable.

2. SOME PREPARATIONS

In this section, we shall review some crucial definitions, starting in §2.1 with the construction of the Gromov-Witten pseudocycle in both the semipositive and general cases. In §2.2, we will also prove a simple result about Donaldson hypersurfaces that is needed to carry out our application to contact hypersurfaces in the non-semipositive case.

2.1. **Defining the Gromov-Witten pseudocycle.** We will now review the definitions of the moduli spaces that determine the pseudocycle (1.2). We begin with the semipositive case in §2.1.1 before addressing the general case in §2.1.2.

2.1.1. *The semipositive case.* Recall that a closed $2n$-dimensional symplectic manifold (M, ω) is called **semipositive** if there are no spherical homology classes $A \in \pi_2(M)$ satisfying

$$\omega(A) > 0 \quad \text{and} \quad 3 - n \leq c_1(A) < 0.$$

In particular, this is always satisfied if $n = 2$ or 3. Under this condition, one can define integer-valued Gromov-Witten invariants

$$\mathrm{GW}_{0,m,A}^{(M,\omega)} : H^*(M; \mathbb{Q})^{\otimes m} \to \mathbb{Z}$$

for any $m \geq 3$ and $A \in H_2(M)$ by the following prescription explained in [MS04]. (The original construction of these invariants is due to Ruan [Rua96].)

Let $\mathcal{J}_\tau(M, \omega)$ denote the space of smooth ω-tame almost complex structures on M, and define

$$\mathcal{J}_{S^2} := \left\{ J \in \Gamma(\mathrm{pr}_2^* \mathrm{End}_{\mathbb{R}}(TM)) \mid J(z, \cdot) \in \mathcal{J}_\tau(M, \omega) \text{ for all } z \in S^2 \right\},$$

where $\mathrm{pr}_2 : S^2 \times M \to M$ denotes the projection. We call $\mathcal{J}_{S^2}$ the space of smooth ω-tame *domain-dependent* almost complex structures (where the "domain" is S^2). Given $J \in \mathcal{J}_{S^2}$, a smooth map $u : S^2 \to M$ is said to be **J-holomorphic** if for all $z \in S^2$,

$$(2.1) \qquad du(z) + J(z, u(z)) \circ du(z) \circ i = 0,$$

where i is the standard complex structure on $S^2 = \mathbb{C} \cup \{\infty\}$. For any $m \geq 3$ and $A \in H_2(M)$, we can then define the moduli space

$$\mathcal{M}_{0,m}^A(M, J) = \{(u, \mathbf{z})\},$$

where $u : S^2 \to M$ is a J-holomorphic map with $[u] = A$, and $\mathbf{z} = (z_4, \ldots, z_m)$ is an ordered $(m-3)$-tuple of pairwise distinct points in $S^2 \setminus \{0, 1, \infty\}$. Setting $(z_1, z_2, z_3) := (0, 1, \infty)$, the **evaluation map** is then defined by

$$\mathrm{ev} = (\mathrm{ev}_1, \ldots, \mathrm{ev}_m) : \mathcal{M}_{0,m}^A(M, J) \to M^m,$$
$$\mathrm{ev}_j(u, \mathbf{z}) = u(z_j) \quad \text{for } j = 1, \ldots, m.$$

The **forgetful map** $\Phi : \mathcal{M}_{0,m}^A(M, J) \to \mathcal{M}_{0,m}$ is likewise defined by associating to $(u, \mathbf{z})$ the equivalence class of conformal structures on S^2 with m marked points positioned at $(0, 1, \infty, z_4, \ldots, z_m)$. Note that since we have fixed the positions of the first three marked points, there is no need to divide out reparametrizations.

Under the semipositivity condition, one can show using standard index computations (see [MS04]) that $\mathrm{ev} : \mathcal{M}_{0,m}^A(M, J) \to M^m$ is a pseudocycle of dimension $2(n-3) + 2c_1(A) + 2m$ for generic choices of $J \in \mathcal{J}_{S^2}$, and for such choices, the corresponding Gromov-Witten invariant (without coupling to gravity) can be computed for $\alpha_1, \ldots, \alpha_m \in H^*(M; \mathbb{Z})$ as

$$(2.2) \qquad \mathrm{GW}_{0,m,A}^{(M,\omega)}(\alpha_1, \ldots, \alpha_m) = [\mathrm{ev}] \cdot (\mathrm{PD}(\alpha_1) \times \ldots \times \mathrm{PD}(\alpha_m)) \in \mathbb{Z}.$$

As mentioned already in Remark 1.4, the forgetful map is generally not a pseudocycle for this definition of the moduli space, and we shall therefore ignore coupling to gravity in our discussion of the semipositive case.

The genericity requirement in (2.2) implies that one cannot generally assume J to be domain-independent. It will be important for our application however that one can do the next best thing: fix any $J_1 \in \mathcal{J}_\tau(M, \omega)$, which we shall refer to henceforward as the *reference* almost complex structure. We can regard J_1 as an element of $\mathcal{J}_{S^2}$ with constant dependence on $z \in S^2$, and the tangent space at J_1 to the Fréchet manifold $\mathcal{J}_{S^2}$ is then

$$T_{J_1}\mathcal{J}_{S^2} = \big\{ Y \in \Gamma(\mathrm{pr}_2^* \mathrm{End}_\mathbb{R}(TM)) \mid Y(z,p)J_1(p) + J_1(p)Y(z,p) = 0$$
$$\text{for all } (z,p) \in S^2 \times M \big\}.$$

After choosing a smooth family of metrics on the manifolds of complex structures at points in M, we can write any $J \in \mathcal{J}_{S^2}$ in some C^0-small neighborhood of J_1 as $J(z,p) = \exp_{J_1(p)} Y(z,p)$ for some C^0-small section $Y \in T_{J_1}\mathcal{J}_{S^2}$. Genericity then allows us to conclude the following:

Lemma 2.1. *There exists a sequence $Y_k \in T_{J_1}\mathcal{J}_{S^2}$ converging to 0 in C^∞ such that (2.2) holds with the Gromov-Witten pseudocycle $\mathrm{ev} : \mathcal{M}_{0,m}^A(M, J) \to M^m$ defined for any $J = \exp_{J_1} Y_k$.* $\qquad\square$

2.1.2. *The Cieliebak-Mohnke approach.* We now consider (M, ω) to be an arbitrary closed $2n$-dimensional symplectic manifold that satisfies $[\omega] \in H^2(M; \mathbb{Z})$ but is not necessarily semipositive. The purpose of this section is to summarize the relevant details of the recipe from [CM07] for defining the Gromov-Witten invariants.

As auxiliary data, we choose an ω-compatible almost complex structure J_0, and a so-called *Donaldson hypersurface of degree $D \in \mathbb{N}$*:

$$Z_D \subset (M, \omega) \text{ symplectic, such that } \mathrm{PD}[Z_D] = D[\omega].$$

The existence of Z_D for large $D \gg 0$ is provided by a deep theorem of Donaldson [Don96], and we can assume moreover that Z_D is *nearly J_0-holomorphic*, in the sense that its *Kähler angle* (see [Don96, p. 669]) is arbitrarily small if D is sufficiently large. It follows in particular that for any $\epsilon > 0$, if $D > 0$ is sufficiently large, one can find $J_1 \in \mathcal{J}_\tau(M, \omega)$ with $\|J_1 - J_0\|_{C^0} < \epsilon$ such that Z_D is J_1-holomorphic. We shall assume in the following that such a $J_1 \in \mathcal{J}_\tau(M, \omega)$ has been chosen and is fixed.

For an integer $k \geq 0$, suppose T is a k-**labelled tree**, i.e. a tree together with a partition of $\{1, \ldots, k\}$ assigning some subset to each vertex $\alpha \in T$. We shall write $\alpha E \beta$ whenever T contains an edge connecting the vertices $\alpha, \beta \in T$, and denote by $\alpha_j \in T$ the vertex associated to $j \in \{1, \ldots, k\}$ by the labelling. Then if S_α denotes a copy of S^2 for each $\alpha \in T$, we can regard a **nodal curve** with k marked points **modelled on** T as a tuple

$$\mathbf{z} = \big(\{z_{\alpha\beta} \in S_\alpha\}_{\alpha E \beta}, \{z_j \in S_{\alpha_j}\}_{j \in \{1, \ldots, k\}} \big)$$

such that for each $\alpha \in T$, all the points in this tuple lying on S_α (the **special points**) are distinct. We associate to $\mathbf{z}$ the *nodal Riemann surface*

$$\Sigma_{\mathbf{z}} := \coprod_{\alpha \in T} S_\alpha \Big/ z_{\alpha\beta} \sim z_{\beta\alpha},$$

where each component S_α is assumed to carry the standard complex structure i. The nodal curve $\mathbf{z}$ (or equivalently the nodal Riemann surface $\Sigma_{\mathbf{z}}$) is called **stable** if for each vertex $\alpha \in T$, there are at least three special

points; note that this is actually a property of the labelled tree T, so we can equivalently say $\mathbf{z}$ is stable if it is modelled on a **stable k-labelled tree**. In this case, $\mathbf{z}$ represents an element $[\mathbf{z}]$ of the Deligne-Mumford space $\overline{\mathcal{M}}_{0,k}$. There is a natural **stabilization** map $\mathbf{z} \mapsto \mathrm{st}(\mathbf{z})$ that makes any nodal curve $\mathbf{z}$ into a stable nodal curve $\mathrm{st}(\mathbf{z})$ by removing vertices with fewer than three special points and placing marked points on neighboring vertices as necessary; this determines a holomorphic surjection on the corresponding nodal Riemann surfaces

$$\mathrm{st} : \Sigma_{\mathbf{z}} \to \Sigma_{\mathrm{st}(\mathbf{z})}.$$

For each $\alpha \in T$, denote by $\mathcal{J}_{S_\alpha}$ a copy of the space $\mathcal{J}_{S^2}$ of domain-dependent almost complex structures defined in the previous section, and let

$$\mathcal{J}_T := \prod_{\alpha \in T} \mathcal{J}_{S_\alpha}.$$

For $J \in \mathcal{J}_T$, a **nodal J-holomorphic map with k marked points** is a pair $(\mathbf{z}, \mathbf{u})$, where $\mathbf{z}$ is a nodal curve with k marked points modelled on T, and $\mathbf{u} : \Sigma_{\mathbf{z}} \to M$ is a continuous map whose restriction to each sphere $S_\alpha \subset \Sigma_{\mathbf{z}}$ is smooth and J-holomorphic (in the sense of (2.1)) with respect to the S_α-dependent almost complex structure determined by J.

Recall next that since $\overline{\mathcal{M}}_{0,k+1}$ is a smooth manifold for any $k \geq 2$, we can consider $\overline{\mathcal{M}}_{0,k+1}$-dependent almost complex structures

$$J \in \Gamma(\mathrm{pr}_2^* \mathrm{End}_{\mathbb{R}}(TM)) \quad \text{such that} \quad J([\mathbf{z}], \cdot) \in \mathcal{J}_\tau(M, \omega),$$

where as usual we denote the projection $\mathrm{pr}_2 : \overline{\mathcal{M}}_{0,k+1} \times M \to M$. For $k \geq 3$, this has a convenient interpretation using the canonical projection

$$\pi : \overline{\mathcal{M}}_{0,k+1} \to \overline{\mathcal{M}}_{0,k}$$

which forgets the last marked point and stabilizes the result. Namely, for any nodal curve $\mathbf{z}$ with k marked points, $\pi^{-1}([\mathrm{st}(\mathbf{z})])$ can be identified canonically with the nodal curve $\Sigma_{\mathrm{st}(\mathbf{z})}$, i.e. we parametrize $\pi^{-1}([\mathrm{st}(\mathbf{z})])$ via the position of the extra marked point. Thus if $\mathbf{z}$ is modelled on the k-labelled tree T, we can associate to $\mathbf{z}$ and the family J above a $\Sigma_{\mathbf{z}}$-dependent almost complex structure

$$J_{\mathbf{z}} \in \mathcal{J}_T, \qquad J_{\mathbf{z}}(z, \cdot) := J([\mathrm{st}(\mathbf{z}), \mathrm{st}(z)], \cdot),$$

where we use $[\mathrm{st}(\mathbf{z}), \mathrm{st}(z)]$ as shorthand for the element of $\pi^{-1}([\mathrm{st}(\mathbf{z})]) \in \overline{\mathcal{M}}_{k+1}$ corresponding to $\mathrm{st}(z) \in \Sigma_{\mathrm{st}(\mathbf{z})}$ under the above identification. For technical reasons, it is important to consider only families J that are **coherent** in the sense defined in [CM07, §3], and we shall denote the space of smooth $\overline{\mathcal{M}}_{0,k+1}$-dependent ω-tame almost complex structures satisfying this condition by

$$\mathcal{J}_{k+1} = \left\{ J : \overline{\mathcal{M}}_{0,k+1} \to \mathcal{J}_\tau(M, \omega) \mid J \text{ is coherent} \right\}.$$

For our purposes, all that we will need to know about the coherence condition is stated in the following lemma, which follows immediately from the definition in [CM07, §3].

Lemma 2.2. *For any $J \in \mathcal{J}_{k+1}$, if $\mathbf{z}$ is a nodal curve modelled on the k-labelled tree T, then for each $\alpha \in T$, the restriction of the family*

$$\Sigma_{\mathbf{z}} \to \mathcal{J}_\tau(M, \omega) : z \mapsto J_{\mathbf{z}}(z, \cdot)$$

to S_α depends only on $z \in S_\alpha$ and the special points of $\mathbf{z}$ on S_α. $\square$

We can now define the moduli spaces needed for the Gromov-Witten invariants. Given an integer $m \geq 0$ and $A \in H_2(M)$, let

$$\ell := A \cdot [Z_D] = D\omega(A) \in \mathbb{N}.$$

We may easily assume $\ell > 3$ by making $D \in \mathbb{N}$ sufficiently large (in general it will be much larger). Choose $J \in \mathcal{J}_{\ell+1}$ with the property that

$$J([\mathbf{z}], \cdot) \equiv J_1 \quad \text{in a neighborhood of } Z_D, \text{ for all } [\mathbf{z}] \in \overline{\mathcal{M}}_{0,\ell+1}.$$

Using the canonical projection $\pi_m : \overline{\mathcal{M}}_{0,m+\ell+1} \to \overline{\mathcal{M}}_{0,\ell+1}$ that forgets the first m marked points and then stabilizes, we can associate to J a coherent $\overline{\mathcal{M}}_{0,m+\ell+1}$-dependent almost complex structure $\pi_m^* J$. Then for any nodal curve $\mathbf{z}$ modelled on an $(m+\ell)$-labelled tree T, we regard a map $\mathbf{u} : \Sigma_{\mathbf{z}} \to M$ as J-holomorphic if it satisfies the Cauchy-Riemann equation (2.1) for the $\Sigma_{\mathbf{z}}$-dependent almost complex structure $(\pi_m^* J)_{\mathbf{z}}$. Given homology classes

$$\{A_\alpha \in H_2(M)\}_{\alpha \in T} \quad \text{such that} \quad \sum_{\alpha \in T} A_\alpha = A,$$

the pair $(T, \{A_\alpha\})$ is called a **weighted tree**, and it is called **stable** if every vertex $\alpha \in T$ with $A_\alpha = 0$ has at least three *special points*, i.e. marked points plus adjacent vertices. We define $\widetilde{\mathcal{M}}_T^{\{A_\alpha\}}(M, J; Z_D)$ to be the space of pairs $(\mathbf{z}, \mathbf{u})$ as above such that $[\mathbf{u}|_{S_\alpha}] = A_\alpha$ for each $\alpha \in T$ and $\mathbf{u}$ maps each of the last ℓ marked points into Z_D. Note that since Z_D is J-holomorphic (as J matches J_1 near Z_D), all isolated intersections of $\mathbf{u}$ with Z_D are positive; in particular, whenever $\mathbf{z}$ has no nodes and $A \neq 0$, the relation $\ell = A \cdot [Z_D]$ implies that either the image of $\mathbf{u}$ is contained in Z_D or the intersections of $\mathbf{u}$ with Z_D occur *only* at the last ℓ marked points. The former is excluded under suitable assumptions on J and for sufficiently large $D \in \mathbb{N}$, due to [CM07, Propositions 8.13 and 8.14].

Remark 2.3. The class of holomorphic curves defined above has the crucial property that *all* isolated intersections with Z_D are positive, not only the guaranteed intersections at the last ℓ marked points. Since the count of these intersections is controlled topologically, positivity provides the necessary lower bound on the number of marked points on components of nodal curves, guaranteeing that such curves have stable domains (see [CM07] for details).

We write $(\mathbf{z}, \mathbf{u}) \sim (\mathbf{z}', \mathbf{u}')$ if there exists a biholomorphic isomorphism between the nodal curves $\mathbf{z}$ and $\mathbf{z}'$ such that $\mathbf{u}$ and $\mathbf{u}'$ are correspondingly related by reparametrization. We then define the moduli space of **J-holomorphic curves modelled on** $(T, \{A_\alpha\})$ as

$$\mathcal{M}_T^{\{A_\alpha\}}(M, J; Z_D) = \widetilde{\mathcal{M}}_T^{\{A_\alpha\}}(M, J; Z_D)\big/ \sim,$$

along with the **evaluation map**,

$$\mathrm{ev} = (\mathrm{ev}_1, \dots, \mathrm{ev}_m) : \mathcal{M}_T^{\{A_\alpha\}}(M, J; Z_D) \to M^m,$$

which evaluates $\mathbf{u}$ at its first m marked points. If $m \geq 3$, we can also define the **forgetful map**

$$\Phi : \mathcal{M}_T^{\{A_\alpha\}}(M, J; Z_D) \to \overline{\mathcal{M}}_{0,m},$$

which forgets both the map $\mathbf{u}$ and the last ℓ marked points of $\mathbf{z}$, and then stabilizes the resulting nodal curve with m marked points. The *top stratum* is the component

$$\mathcal{M}^A_{0,m+\ell}(M, J; Z_D) := \mathcal{M}^{\{A_\alpha\}}_T(M, J; Z_D), \text{ where } |T| = 1,$$

consisting of equivalence classes $[(\mathbf{z}, \mathbf{u})]$ such that $\mathbf{z}$ has no nodes; in this case $\mathbf{u} : S^2 \to M$ is simply a pseudoholomorphic sphere, for some domain-dependent almost complex structure determined by J and the positions of its last ℓ marked points. The union of the spaces $\mathcal{M}^{\{A_\alpha\}}_T(M, J; Z_D)$ for all stable weighted trees $(T, \{A_\alpha\})$ with $\sum_\alpha A_\alpha = A$ carries a natural topology as a metrizable Hausdorff space, the *Gromov topology*, and we denote by

$$\overline{\mathcal{M}}^A_{0,m+\ell}(M, J; Z_D) \subset \bigcup_{(T, \{A_\alpha\}) \text{ stable}} \mathcal{M}^{\{A_\alpha\}}_T(M, J; Z_D)$$

the closure of $\mathcal{M}^A_{0,m+\ell}(M, J; Z_D)$ in this space.

If $m \geq 3$, then for suitable choices of $J \in \mathcal{J}_{\ell+1}$ matching the reference structure J_1 near Z_D,

(2.3) $$(\mathrm{ev}, \Phi) : \mathcal{M}^A_{0,m+\ell}(M, J; Z_D) \to M^m \times \overline{\mathcal{M}}_{0,m}$$

is a pseudocycle of dimension

$$\dim \mathcal{M}^A_{0,m+\ell}(M, J; Z_D) = 2(n-3) + 2c_1(A) + 2m,$$

and the resulting rational Gromov-Witten invariants

$$\mathrm{GW}^{(M,\omega)}_{0,m,A} : H^*(M; \mathbb{Q})^{\otimes m} \otimes H_*(\overline{\mathcal{M}}_{0,m}; \mathbb{Q}) \to \mathbb{Q},$$

(2.4)
$$\mathrm{GW}^{(M,\omega)}_{0,m,A}(\alpha_1, \ldots, \alpha_m, \beta) = \\ \frac{1}{\ell!}[(\mathrm{ev}, \Phi)] \cdot (\mathrm{PD}(\alpha_1) \times \ldots \times \mathrm{PD}(\alpha_m) \times \beta)$$

are independent of all choices. If one excludes the forgetful map and $\beta \in H_*\left(\overline{\mathcal{M}}_{0,m}\right)$ from this statement, then it is simply the main result of [CM07] (and is also valid for any $m \geq 0$). We will explain in Appendix A how the arguments of Cieliebak and Mohnke can be modified to include the forgetful map in the discussion.

As alluded to above, the constructions in [CM07] require some extra assumptions on $J \in \mathcal{J}_{\ell+1}$ in order to define the Gromov-Witten invariants, but the details of these assumptions will not concern us beyond the following analogue of Lemma 2.1. Recall that we have fixed a reference almost complex structure J_1 for which the Donaldson hypersurface Z_D is J_1-holomorphic. We can trivially regard J_1 as an element of $\mathcal{J}_{\ell+1}$ with constant dependence on $\overline{\mathcal{M}}_{0,\ell+1}$. Then any other element of $\mathcal{J}_{\ell+1}$ that is C^0-close to J_1 can be written as

$$J = \exp_{J_1} Y$$

for some $Y \in T_{J_1}\mathcal{J}_{\ell+1}$, where the latter is the Fréchet space of *coherent* (see [CM07, §3]) smooth sections of $\mathrm{pr}_2^* \mathrm{End}_{\mathbb{R}}(TM) \to \overline{\mathcal{M}}_{0,\ell+1} \times M$ satisfying

$$Y([\mathbf{z}], p)J_1(p) + J_1(p)Y([\mathbf{z}], p) = 0 \quad \text{ for all } ([\mathbf{z}], p) \in \overline{\mathcal{M}}_{0,\ell+1} \times M.$$

Lemma 2.4. *There exists a sequence $Y_k \in T_{J_1} \mathcal{J}_{\ell+1}$ converging to 0 in C^∞ such that (2.4) holds with the Gromov-Witten pseudocycle (2.3) defined for any $J = \exp_{J_1} Y_k$.* $\qquad\qquad\square$

2.2. **Donaldson hypersurfaces transverse to a contact hypersurface.**
In order to apply the Gromov-Witten invariants of [CM07] to a situation involving pseudoconvex hypersurfaces, we need the following additional fact about Donaldson hypersurfaces.

Proposition 2.5. *Suppose (M, ω) is a closed $2n$-dimensional symplectic manifold with $[\omega] \in H^2(M; \mathbb{Z})$, J_0 is an ω-compatible almost complex structure, and $V \subset M$ is a closed $(2n-1)$-dimensional J_0-convex hypersurface with induced contact structure*

$$\xi = TV \cap J_0(TV) \subset TV.$$

Then for all $D \in \mathbb{N}$ sufficiently large, there exists a Donaldson hypersurface $Z_D \subset (M, \omega)$ of degree D that intersects V transversely in a contact submanifold of (V, ξ). Moreover, for any $\epsilon > 0$, if $D \in \mathbb{N}$ is sufficiently large, then one can find Z_D with the above property and an ω-tame almost complex structure J_1 on M such that

(1) *Z_D is J_1-holomorphic;*
(2) *V is J_1-convex with $\xi = TV \cap J_1(TV)$;*
(3) *$\|J_1 - J_0\|_{C^0} < \epsilon$.*

The proposition is a straightforward application of Mohsen's relative version [Moh] of an estimated transversality result of Donaldson and Auroux [Don96, Aur97]. To explain this, we must recall some details from the asymptotically holomorphic methods of Donaldson and Auroux, as used by Mohsen.

We first need to define a quantitative measurement of the distance of a real subspace of a complex vector space from being complex. Suppose (E, J) is a finite-dimensional complex vector space with Hermitian inner product g, and write $|v| := \sqrt{g(v, v)}$ for $v \in E$. Then for any real-linear subspace $E' \subset E$ of even dimension, define

$$\Theta_g(E'; E, J) := \max_{v \in E', \ |v|=1} \operatorname{dist}\left(Jv, E'\right)$$

$$= \max_{v \in E', \ |v|=1} \left(\min_{w \in E'} |Jv - w|\right).$$

It will be useful to note that this definition depends on the Hermitian metric only up to positive rescaling, i.e.

(2.5) $$\Theta_{cg}(E'; E, J) = \Theta_g(E'; E, J) \quad \text{for all } c > 0.$$

It also depends continuously on all the data, thus if B is a compact space and $(E, J) \to B$ is a complex vector bundle of finite rank with Hermitian bundle metric g, then for any real subbundle $E' \subset E$ of even rank, we can similarly define

$$\Theta_g(E'; E, J) := \max_{p \in B} \Theta_g(E_p'; E_p, J) \geq 0.$$

Observe that if ω is any symplectic structure on (E, J) that tames J, then any sufficiently small perturbation of a complex subbundle is automatically also a symplectic subbundle, thus we have the following.

Lemma 2.6. *Suppose B is a compact space and $(E, J) \to B$ is a complex vector bundle of finite rank, equipped with a Hermitian bundle metric g. In each of the following statements, assume $E' \subset E$ is a real subbundle of even rank.*

(a) *E' is a complex subbundle of (E, J) if and only if $\Theta_g(E'; E, J) = 0$.*
(b) *For any C^0-open neighborhood $\mathcal{U}_J$ of J in the space of smooth complex structures on E, there exists a number $c > 0$ such that every $E' \subset E$ with $\Theta_g(E'; E, J) < c$ is a complex subbundle of (E, J') for some $J' \in \mathcal{U}_J$.*
(c) *For any symplectic structure ω on $E \to B$ that tames J, there exists a number $c' > 0$ such that every $E' \subset E$ satisfying $\Theta_g(E'; E, J) < c'$ is a symplectic subbundle of (E, ω).*

$\square$

In order to relate the above definition to questions of estimated transversality, we define (following [Moh]) for any real-linear map $A : V \to W$ between finite-dimensional Euclidean vector spaces, the **surjectivity modulus**

$$\mathrm{Surj}(A) := \min_{\lambda \in W^* \backslash \{0\}} \frac{\|\lambda \circ A\|}{\|\lambda\|} \geq 0.$$

Lemma 2.7. *The surjectivity modulus has the following properties.*

(a) *$\mathrm{Surj}(A) > 0$ if and only if A is surjective, and in this case*

$$\mathrm{Surj}(A) \geq \sup \left\{ \frac{1}{\|B\|} \ \middle| \ B : W \to V \text{ is a right inverse of } A \right\}.$$

(b) *For any two real-linear maps $A, B : V \to W$,*

$$\mathrm{Surj}(A + B) \geq \mathrm{Surj}(A) - \|B\|.$$

(c) *Suppose (V, J, g) and (V', J', g') are finite-dimensional Hermitian vector spaces and $A = A^{1,0} + A^{0,1} : V \to V'$ is real-linear, where $A^{1,0}$ and $A^{0,1}$ denote the complex linear and antilinear parts respectively. Then*

(2.6) $$\Theta_g(\ker A; V, J) \leq 2 \frac{\|A^{0,1}\|}{\mathrm{Surj}(A)}.$$

Proof. The first two properties are proved by straightforward computations. The following proof of the third property was explained to me by Jean-Paul Mohsen.

Let $V^*_{\ker A} = \{\mu \in V^* \mid \mu|_{\ker A} = 0\}$, which is precisely the space of dual vectors on V of the form $\{\mu = \lambda \circ A \in V^* \mid \lambda \in W^*\}$. Now suppose $v \in \ker A$ and $|v| = 1$. The distance of Jv from $\ker A$ is the norm of its second part

under the orthogonal decomposition $V = (\ker A) \oplus (\ker A)^\perp$, hence

$$\operatorname{dist}(Jv, \ker A) = \max_{w \in (\ker A)^\perp \setminus \{0\}} \frac{|\langle w, Jv \rangle|}{|w|} = \max_{\mu \in V^*_{\ker A} \setminus \{0\}} \frac{|\mu(Jv)|}{\|\mu\|}$$
$$= \max_{\lambda \in W^* \setminus \{0\}} \frac{|\lambda \circ A(Jv)|}{\|\lambda \circ A\|}.$$

Now, using the fact that $Av = 0$ and that $A^{1,0}$ commutes while $A^{0,1}$ anti-commutes with the complex structures, we have

$$A(Jv) = A^{1,0}Jv + A^{0,1}Jv = J'A^{1,0}v - J'A^{0,1}v = -2J'A^{0,1}v,$$

hence $|A(Jv)| \le 2\|A^{0,1}\|$, implying

$$\operatorname{dist}(Jv, \ker A) \le \max_{\lambda \in W^* \setminus \{0\}} \frac{2\|\lambda\| \cdot \|A^{0,1}\|}{\|\lambda \circ A\|} = 2\frac{\|A^{0,1}\|}{\operatorname{Surj}(A)}.$$

$$\square$$

Next, assume (M, ω) is a closed symplectic manifold with $[\omega] \in H^2(M; \mathbb{Z})$, and J_0 is an ω-compatible almost complex structure. This determines the sequence of Riemannian metrics

$$g := \omega(\cdot, J\cdot), \qquad g_D := D \cdot g \text{ for } D \in \mathbb{N}$$

on M. Let $L \to M$ denote a complex line bundle with $c_1(L) = [\omega]$, equipped with a Hermitian metric $\langle \, , \, \rangle$ and a Hermitian connection ∇ whose curvature 2-form is $-2\pi i\omega$. For $D \in \mathbb{N}$, we also consider the D-fold tensor power $L^{\otimes D} \to M$, with its induced Hermitian metric and Hermitian connection, also denoted by $\langle \, , \, \rangle$ and ∇ respectively; the latter has curvature $-2\pi iD\omega$. For sections $s : M \to L^{\otimes D}$, we denote by ∂s and $\bar{\partial}s$ respectively the complex linear and antilinear parts of the covariant derivative ∇s. We will always define C^0-norms of ∇s and related tensors with respect to the metrics g_D on TM and $\langle \, , \, \rangle$ on $L^{\otimes D}$, e.g.

$$\|\nabla s(p)\|_{g_D} := \max_{X \in T_pM \setminus \{0\}} \frac{|\nabla_X s|}{|X|_{g_D}} \qquad \text{for } p \in M,$$
$$\|\nabla s\|_{g_D} := \sup_{p \in M} \|\nabla s(p)\|_{g_D},$$

where $|X|_{g_D} := \sqrt{g_D(X, X)}$ for $X \in T_pX$ and $|v| := \sqrt{\langle v, v \rangle}$ for $v \in L_p^{\otimes D}$. The surjectivity modulus of $\nabla s(p)$ at points $p \in M$ will also be defined relative to this choice of metrics, which we shall indicate via the notation

$$\operatorname{Surj}_{g_D}(\nabla s(p)) := \min_{0 \ne \lambda \in \operatorname{Hom}_{\mathbb{R}}\left(L_p^{\otimes D}, \mathbb{R}\right)} \frac{\|\lambda \circ \nabla s(p)\|_{g_D}}{\|\lambda\|}.$$

This means $\operatorname{Surj}_{g_D}(\nabla s(p)) = \frac{1}{\sqrt{D}} \operatorname{Surj}_g(\nabla s(p))$.

The next two definitions are essentially due to Auroux [Aur97], though we have made minor modifications to fit them into the framework of [Moh].

Definition 2.8. Given constants $C > 0$ and $r \in \mathbb{N}$, we say that a sequence of sections $s_D : M \to L^{\otimes D}$ (for large $D \in \mathbb{N}$) is C-**asymptotically holomorphic up to order** $r \in \mathbb{N}$ if for all D sufficiently large,

$$(2.7) \qquad \|s_D\|_{g_D} \leq C, \qquad \|\nabla^m s_D\|_{g_D} \leq C, \qquad \|\nabla^{m-1}\bar{\partial}s_D\|_{g_D} \leq \frac{C}{\sqrt{D}}$$

$$\text{for each } m = 1, \ldots, r.$$

Definition 2.9. Given a constant $\eta > 0$ and a submanifold $V \subset M$, we say that a sequence of sections $s_D : M \to L^{\otimes D}$ (for large $D \in \mathbb{N}$) is η-**transverse along** V if for all sufficiently large D,

$$|s_D(p)| < \eta \quad \Rightarrow \quad \text{Surj}_{g_D}\left(\nabla s_D(p)|_{T_p V}\right) \geq \eta \qquad \text{for all } p \in V.$$

For any (M, ω) and J_0 as above, Donaldson [Don96] constructs a sequence of sections $s_D : M \to L^{\otimes D}$ that are, for some $K, \eta > 0$, K-asymptotically holomorphic up to order 2 and globally η-transverse (i.e. η-transverse along M). It follows via (2.5) and Lemma 2.7(c) that for sufficiently large $D \in \mathbb{N}$, $Z_D := s_D^{-1}(0) \subset M$ are smooth submanifolds with

$$\Theta_g(TZ_D; TM|_{Z_D}, J_0) = \Theta_{g_D}(TZ_D; TM|_{Z_D}, J_0)$$

$$\leq \max_{p \in Z_D} \frac{2\|\bar{\partial}s_D(p)\|_{g_D}}{\text{Surj}_{g_D}\left(\nabla s(p)\right)}$$

$$\leq 2\frac{K/\sqrt{D}}{\eta} \to 0 \quad \text{as } D \to \infty.$$

Thus by Lemma 2.6, the submanifolds $Z_D \subset (M, \omega)$ are symplectic and uniformly close to being J_0-holomorphic for sufficiently large D. These are the Donaldson hypersurfaces that we made use of in the previous section; indeed, they satisfy $\text{PD}[Z_D] = c_1(L^{\otimes D}) = Dc_1(L) = D[\omega] \in H^2(M)$.

For our purposes, the relevant case of Mohsen's extension of the Donaldson-Auroux transversality theorem can now be stated as follows.

Proposition 2.10 ([Moh, Théorème 2.2])**.** *Assume (M, ω) is a closed $2n$-dimensional symplectic manifold with an ω-compatible almost complex structure J_0, $V \subset M$ is a closed submanifold of dimension $2n - 1$, and $\xi \subset TV$ denotes the J_0-complex subbundle*

$$\xi := TV \cap J_0(TV).$$

Then given any $K > 0$, $\epsilon > 0$ and $m_{\max} \in \mathbb{N}$, there exist $D_0 \in \mathbb{N}$ and $\eta > 0$ such that the following holds. For any sequence of sections $s_D : M \to L^{\otimes D}$ (for large D) which are K-asymptotically holomorphic up to order 2, there exists a sequence (for large D) of sections $t_D : M \to L^{\otimes D}$ such that, for all $D \geq D_0$, the sequence t_D is ϵ-asymptotically holomorphic up to order $m_{\max}$, and the sequence $s'_D := s_D + t_D$ is η-transverse along V, and also satisfies

$$p \in V \text{ and } |s'_D(p)| < \eta \quad \Rightarrow \quad \text{Surj}_{g_D}\left(\nabla s'_D(p)|_{\xi_p}\right) \geq \eta.$$

$\square$

Proof of Proposition 2.5. Assume $V \subset M$ is J_0-convex, and let $s_D : M \to L^{\otimes D}$ denote the K-asymptotically holomorphic and globally η-transverse sequence of sections provided by [Don96]. Pick $\epsilon \in (0, \eta)$, and let $t_D : M \to L^{\otimes D}$ denote the ϵ-asymptotically holomorphic sequence provided by

Proposition 2.10, giving rise to the perturbed sections $s_D' := s_D + t_D$ and zero-sets $Z_D := (s_D')^{-1}(0) \subset M$. Using Lemma 2.7(b), we may assume s_D' is also K-asymptotically holomorphic and η-transverse after making the substitutions $K \mapsto K + \epsilon > 0$ and $\eta \mapsto \eta - \epsilon > 0$, and by shrinking $\eta > 0$ further if necessary, Proposition 2.10 also guarantees

$$\mathrm{Surj}_{g_D} \left(\nabla s_D'(p)|_{\xi_p} \right) \geq \eta$$

for all $p \in Z_D \cap V$. This implies that for sufficiently large D, $Z_D \subset (M, \omega)$ is a symplectic submanifold and intersects both V and the distribution $\xi \subset TV$ transversely, hence the submanifold

$$\Sigma_D := Z_D \cap V \subset V$$

inherits a smooth oriented hyperplane bundle

$$\xi_D := TZ_D \cap \xi \subset T\Sigma_D.$$

Regarding ξ_D as a real subbundle of the complex vector bundle $(\xi|_{\Sigma_D}, J_0)$, Lemma 2.7(c) and (2.5) now imply

$$\Theta_g \left(\xi_D; \xi|_{\Sigma_D}, J_0 \right) \leq \max_{p \in \Sigma_D} \frac{2\|\bar{\partial}s_D'(p)|_{\xi_p}\|_{g_D}}{\mathrm{Surj}_{g_D} \left(\nabla s_D'(p)|_{\xi_p} \right)} \leq \frac{2K}{\eta\sqrt{D}} \to 0$$

as $D \to \infty$. Since V is J_0-convex, there exists a contact form α on V such that $\xi = \ker \alpha$ and $d\alpha|_\xi$ is a symplectic vector bundle structure that tames J_0. Applying Lemma 2.6, we therefore conclude from the above that $(\xi_D, d\alpha)$ is a symplectic subbundle of $(\xi|_{\Sigma_D}, d\alpha)$ for sufficiently large D, implying that $\alpha|_{T\Sigma_D}$ is contact, so $\Sigma_D \subset (V, \xi)$ is a contact submanifold. Moreover, the complex structure $J_0|_\xi$ along Σ_D admits a C^0-small perturbation to a complex structure J_1 on ξ along Σ_D for which ξ_D is J_1-invariant. Following the extension procedure of [CM07, §8], J_1 can then be extended to an almost complex structure on M that preserves ξ along V, preserves TZ_D and is C^0-close to J_0 for sufficiently large D. Note that having J_1 be C^0-close to J_0 implies that $J_1|_\xi$ is also tamed by $d\alpha|_\xi$ without loss of generality, thus V is J_1-convex.   □

3. THE PROOF

We now proceed to the proof of the main theorem.

Suppose (M, ω) is a closed and connected symplectic manifold with an almost complex structure J such that either of the following conditions are satisfied:

- (M, ω) is semipositive and J is ω-tame;
- $[\omega] \in H^2(M; \mathbb{Z})$ and J is ω-compatible.

We will assume the Gromov-Witten invariants to be defined via the prescriptions in §2.1.1 or §2.1.2 accordingly. Suppose $V \subset M$ is a J-convex hypersurface. Arguing by contradiction, we assume there is a nontrivial Gromov-Witten invariant of the form

(3.1) $\mathrm{GW}_{0,m,A}^{(M,\omega)}(\mathrm{PD}[V] \cup \alpha_1, \alpha_2, \ldots, \alpha_m; \beta) \neq 0$

for some $m \geq 3$, $A \in H_2(M)$, $\alpha_1, \ldots, \alpha_m \in H^*(M; \mathbb{Q})$ and $\beta \in H_*(\overline{\mathcal{M}}_{0,m}; \mathbb{Q})$. The essential idea of the proof will be show that (3.1) implies the existence

of a pseudoholomorphic sphere that touches V tangentially from the wrong side, thus contradicting pseudoconvexity.

Remark 3.1. In the following we will give a unified argument that applies to both the semipositive and non-semipositive cases, referring as necessary to the slightly different sets of definitions in §2.1.1 and §2.1.2. For the semipositive case, some statements would need to be modified in obvious ways by removing all references to $\beta \in H_*(\overline{\mathcal{M}}_{0,m})$ and the forgetful map (see Remark 1.4).

We must now choose a perturbed almost complex structure J_1 that is suitably adapted to the definition of the Gromov-Witten invariants. In the semipositive case, it suffices to set $J_1 = J$. If (M, ω) is not semipositive, then we have assumed $[\omega] \in H^2(M; \mathbb{Z})$ and can therefore find a sequence of Donaldson hypersurfaces Z_D of large degrees $D \in \mathbb{N}$ as described in §2.1.2. By Proposition 2.5, after making the degree sufficiently large, we can find a smooth ω-tame almost complex structure J_1 that is arbitrarily C^0-close to J while making Z_D a J_1-holomorphic hypersurface and V simultaneously a J_1-convex hypersurface. We shall treat J_1 as the *reference* almost complex structure used in Lemmas 2.1 and 2.4.

Let J' denote a generic domain-dependent or $\overline{\mathcal{M}}_{\ell+1}$-dependent perturbation of J_1 as described in §2.1.1 or §2.1.2 respectively, giving rise to the moduli space $\mathcal{M}_{0,m}^A(M, J')$ of J'-holomorphic spheres homologous to A, with the associated evaluation/forgetful pseudocycle

$$(\mathrm{ev}, \Phi) = (\mathrm{ev}_1, \ldots, \mathrm{ev}_m, \Phi) : \mathcal{M}_{0,m}^A(M, J') \to M^m \times \overline{\mathcal{M}}_{0,m}.$$

In the non-semipositive case, we are assuming as in §2.1.2 that J' matches J_1 near Z_D and the elements of $\mathcal{M}_{0,m}^A(M, J')$ have extra marked points constrained to lie in Z_D under evaluation, but these details will play no role in what follows and we will therefore suppress them in the notation. The condition (3.1) now means

$$[(\mathrm{ev}, \Phi)] \cdot \Big(([V] \cdot \mathrm{PD}(\alpha_1)) \times \mathrm{PD}(\alpha_2) \times \ldots \times \mathrm{PD}(\alpha_m) \times \beta \Big) \neq 0.$$

Lemma 3.2. *There exists a smooth loop*

$$\ell : S^1 \to \mathcal{M}_{0,m}^A(M, J')$$

such that $(\mathrm{ev}_1 \circ \ell)_*[S^1] \cdot [V] \neq 0$.

Proof. We lose no generality by supposing that the classes $\alpha_1, \ldots, \alpha_m \in H^*(M; \mathbb{Q})$ and $\beta \in H_*(\overline{\mathcal{M}}_{0,m})$ are each homogeneous, i.e. they have well-defined degrees. By a theorem of Thom [Tho54], there are rational numbers $c_0, \ldots, c_m \neq 0$ and smooth submanifolds $\bar{\alpha}_1, \ldots, \bar{\alpha}_m \subset M$ and $\bar{\beta} \subset \overline{\mathcal{M}}_{0,m}$ such that

$$c_0[\bar{\beta}] = \beta \in H_*(\overline{\mathcal{M}}_{0,m}; \mathbb{Q}),$$
$$c_i[\bar{\alpha}_i] = \mathrm{PD}(\alpha_i) \in H_*(M; \mathbb{Q}) \quad \text{for } i = 1, \ldots, m.$$

We claim that after generic smooth perturbations of these submanifolds, we may assume the pseudocycle (ev, Φ) is weakly transverse to $\bar{\alpha}_1 \times \ldots \times \bar{\alpha}_m \times \bar{\beta}$

in the sense of [MS04, Definition 6.5.10]. Indeed, we can perturb $\bar{\alpha}_1$ such that ev_1 is weakly transverse to $\bar{\alpha}_1$, so by [MS04, Lemma 6.5.14],

$$\mathrm{ev}_2\,|_{\mathrm{ev}_1^{-1}(\bar{\alpha}_1)} : \mathrm{ev}_1^{-1}(\bar{\alpha}_1) \to M$$

is a pseudocycle of dimension $\dim \mathcal{M}_{0,m}^A(M, J') - \deg \alpha_1$. After perturbing $\bar{\alpha}_2$, we may also assume this new pseudocycle is weakly transverse to $\bar{\alpha}_2$, which means $(\mathrm{ev}_1, \mathrm{ev}_2)$ is now weakly transverse to $\bar{\alpha}_1 \times \bar{\alpha}_2$. Repeating this procedure $m+1$ times proves the claim. With this established, we can define the *constrained moduli space*

$$\mathcal{M}' := (\mathrm{ev}, \Phi)^{-1}(\bar{\alpha}_1 \times \ldots \times \bar{\alpha}_m \times \bar{\beta}),$$

so that $(\mathrm{ev}, \Phi)|_{\mathcal{M}'}$ is a 1-dimensional pseudocycle, which means $\mathcal{M}'$ is a *compact* 1-dimensional submanifold of $\mathcal{M}_{0,m}^A(M, J')$. Now choose a generic smooth perturbation V' of $V \subset M$ such that

$$\bar{\alpha}_1 \pitchfork V' \quad \text{and} \quad \mathrm{ev}_1\,|_{\mathcal{M}'} \pitchfork V'.$$

We then have

$$
(3.2) \quad
\begin{aligned}
c_0 \ldots c_m &\Big((\mathrm{ev}_1)_*[\mathcal{M}'] \cdot [V] \Big) = \\
&[(\mathrm{ev}, \Phi)] \cdot \Big(([V] \cdot \mathrm{PD}(\alpha_1)) \times \mathrm{PD}(\alpha_2) \times \ldots \times \mathrm{PD}(\alpha_m) \times \beta \Big) \neq 0.
\end{aligned}
$$

Any connected component of $\mathcal{M}'$ on which the above intersection number is nonzero is then a smooth loop with the stated property. $\qquad\square$

In order to apply this lemma in proving the main result, we shall borrow an idea from [ABW10]. Observe that by (3.1), $[V] \in H_*(M; \mathbb{Q})$ must be nontrivial, so V is nonseparating. One can therefore construct a connected infinite cover of M, defined by cutting M open along V to produce a cobordism with boundary $-V \sqcup V$, and then gluing together an infinite chain of copies $\{M_n\}_{n \in \mathbb{Z}}$ of this cobordism. Denote for each $n \in \mathbb{Z}$ the boundary of the cobordism M_n by

$$\partial M_n = -V_n^- \sqcup V_n^+,$$

then each $V_n^\pm$ has a neighborhood in M_n naturally identified with a suitable half-neighborhood of V in M, and we use these identifications to glue M_n to M_{n+1} along $V_n^+ = V_{n+1}^-$. This produces a smooth, connected and noncompact manifold (see Figure 1)

$$\widetilde{M} = \bigcup_{n \in \mathbb{Z}} M_n,$$

which has a natural smooth covering projection

$$\pi : \widetilde{M} \to M$$

and is separated by infinitely many copies of the hypersurface V, which we shall denote by

$$V_n := V_n^+ \subset \widetilde{M}.$$

Let

$$\widetilde{J}_1 := \pi^* J_1$$

denote the natural lift of the reference almost complex structure J_1 to the cover $\widetilde{M}$, for which the hypersurfaces V_n are all $\widetilde{J}_1$-convex.
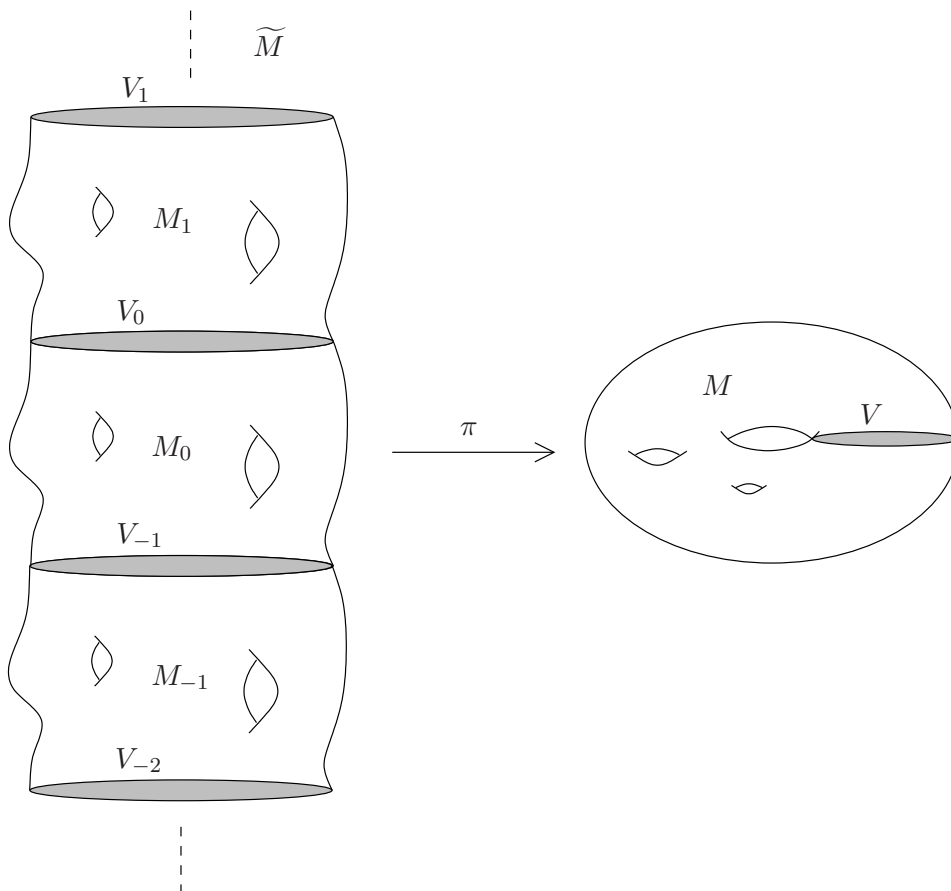
FIGURE 1. The cover $\pi : \widetilde{M} \to M$ defined for a nonseparating hypersurface $V \subset M$.

By Lemma 2.1 or 2.4, we can find a sequence J^k of generic structures for which Lemma 3.2 holds with $J' := J^k$, producing loops

$$\ell_k : S^1 \to \mathcal{M}_{0,m}^A(M, J^k) \quad \text{with} \quad (\mathrm{ev}_1 \circ \ell_k)_*[S^1] \cdot [V] \neq 0 \text{ for all } k,$$

and we may assume moreover that J^k converges in C^∞ as $k \to \infty$ to the domain-independent almost complex structure J_1. For each k and each $\tau \in S^1$, $\ell_k(\tau) \in \mathcal{M}_{0,m}^A(M, J^k)$ is an equivalence class of spheres $u : S^2 \to M$ satisfying a domain-dependent Cauchy-Riemann equation as in (2.1). Since S^2 is simply connected, each of the loops ℓ_k can be lifted to $\widetilde{M}$ as a continuous family of holomorphic spheres $\{u_\tau^k\}_{\tau \in \mathbb{R}}$, and the nontrivial intersection of $\mathrm{ev}_1 \circ \ell_k$ with V implies that evaluation of u_τ^k at the first marked point traces a noncompact path in $\widetilde{M}$ intersecting M_n for every $n \in \mathbb{Z}$. It follows that for each k, there exists a parameter value $\tau_*^k \in \mathbb{R}$ for which the image of $u_{\tau_*^k}^k$ touches V_0 but not the interior of M_1.

We now have a sequence of curves $u^k := u_{\tau_*^k}^k \in \mathcal{M}_{0,m}^A(M, J^k)$ which admit lifts to $\widetilde{M}$ that touch V_0 but not the interior of M_1. This is not yet a contradiction, because the Cauchy-Riemann equation satisfied by each

u^k involves a domain-dependent almost complex structure. As $k \to \infty$, however, Gromov compactness gives a subsequence of u^k converging to a nodal J_1-holomorphic sphere, and at least one component of this nodal curve lifts to a nontrivial $\widetilde{J_1}$-holomorphic sphere in $\widetilde{M}$ that touches V_0 tangentially from below. Since V_0 is a $\widetilde{J_1}$-convex hypersurface, this is a contradiction and thus concludes the proof.

Appendix A. The forgetful map is a pseudocycle

The purpose of this appendix is to justify the statement, made in §2.1.2, that for suitably chosen data, the evaluation/forgetful map

$$(\mathrm{ev}, \Phi) : \mathcal{M}^A_{0,m+\ell}(M, J; Z_D) \to M^m \times \overline{\mathcal{M}}_{0,m}$$

as defined in the setting of Cieliebak and Mohnke [CM07] is a pseudocycle, and its rational cobordism class (after dividing by $\ell!$) is independent of the choices. This is proved in [CM07] for $\mathrm{ev} : \mathcal{M}^A_{0,m+\ell}(M, J; Z_D) \to M^m$, without accounting for the forgetful map, though the arguments necessary for proving the more general statement are almost already present in [CM07], so we shall merely sketch the necessary modifications.

In the following, we will often refer to holomorphic curves that carry distinct sets of **ordinary** and **extra** marked points; for curves in the space $\mathcal{M}^A_{0,m+\ell}(M, J; Z_D)$, this means the first m and last ℓ marked points respectively. Recall that the forgetful map $\Phi : \overline{\mathcal{M}}_{0,m+\ell}(M, J; Z_D) \to \overline{\mathcal{M}}_{0,m}$ is defined by forgetting not only the map into M but also the extra marked points, and then stabilizing.

Remark A.1. Although Φ maps the top stratum $\mathcal{M}^A_{0,m+\ell}(M, J; Z_D)$ into the top stratum $\mathcal{M}_{0,m}$ of $\overline{\mathcal{M}}_{0,m}$, it will not generally define a pseudocycle $\mathcal{M}^A_{0,m+\ell}(M, J; Z_D) \to \mathcal{M}_{0,m}$, mainly because $\mathcal{M}_{0,m}$ itself is not compact.

We assume as in §2.1.2 that J_0 is a compatible almost complex structure on the closed and connected $2n$-dimensional symplectic manifold (M, ω), and $Z_D \subset M$ is a nearly J_0-holomorphic Donaldson hypersurface of large degree $D \in \mathbb{N}$. If D is sufficiently large and $J \in \mathcal{J}_{\ell+1}$ is chosen appropriately (e.g. it must be C^0-close to J_0 and match a reference domain-independent structure J_1 near Z_D, whose restriction to Z_D is generic), then [CM07] shows that the natural compactification $\overline{\mathcal{M}}^A_{0,m+\ell}(M, J; Z_D)$ of $\mathcal{M}^A_{0,m+\ell}(M, J; Z_D)$ consists of strata $\mathcal{M}^{\{A_\alpha\}}_T(M, J; Z_D)$ modelled on weighted $(m + \ell)$-labelled trees $(T, \{A_\alpha\})$ that are ℓ-**stable**, i.e. they are stable even after removing the m ordinary (but keeping the ℓ extra) marked points. Moreover, none of the nonconstant components of such nodal curves are contained in Z_D. The pseudocycle property for (ev, Φ) is based on the observation that on any stratum $\mathcal{M}^{\{A_\alpha\}}_T(M, J; Z_D) \subset \overline{\mathcal{M}}^A_{0,m+\ell}(M, J; Z_D)$ for which T has more than one vertex, the restriction of (ev, Φ) factors as a composition

$$(\text{A.1}) \qquad \mathcal{M}^{\{A_\alpha\}}_T(M, J; Z_D) \to \mathcal{M}^{\{A_\alpha\}}_{T'}(M, J; Z_D) \to M^m \times \overline{\mathcal{M}}_{0,m},$$

where the space in the middle is a smooth manifold that either has dimension at most $\dim \mathcal{M}^A_{0,m+\ell}(M, J; Z_D) - 2$ or factors through another manifold that does. The reason we need this factorization instead of just considering

$\mathcal{M}_T^{\{A_\alpha\}}(M, J; Z_D)$ itself is that the latter sometimes has artificially large dimension, due to the presence of multiple extra marked points in the same constant component. But since these extra marked points play no role in defining the evaluation and forgetful map, we can fix this problem by removing them, which leads to the factorization above. The remainder of this appendix will be concerned with the definition and essential properties of $\mathcal{M}_{T'}^{\{A_\alpha\}}(M, J; Z_D)$.

As in [CM07], we will use the term **ghost tree** to mean a maximal subtree T'' of a weighted tree $(T, \{A_\alpha\})$ with the property that $A_\alpha = 0$ for all $\alpha \in T''$. Similarly, a **ghost bubble** on a nodal J-holomorphic curve $[(\mathbf{z}, \mathbf{u})] \in \mathcal{M}_T^{\{A_\alpha\}}(M, J; Z_D)$ is the constant holomorphic curve obtained by restricting $\mathbf{u}$ to any component $S_\alpha \subset \Sigma_{\mathbf{z}}$ with $A_\alpha = 0$. We shall define the manifold $\mathcal{M}_{T'}^{\{A_\alpha\}}(M, J; Z_D)$ roughly as the space of nodal curves that one obtains from elements of $\mathcal{M}_T^{\{A_\alpha\}}(M, J; Z_D)$ by forgetting all but one of the extra marked points on each ghost tree and stabilizing as necessary, but keeping all other information, including the conformal structures on the ghost bubbles with their ordinary marked points. This can be defined more formally as follows. Suppose $\ell' \leq \ell$ is the number of extra marked points on vertices $\alpha \in T$ with $A_\alpha \neq 0$ plus the number of ghost trees in T that have at least one extra marked point. Then we associate to T a stable $(m + \ell')$-labelled tree T' via the following procedure:

(1) On each ghost tree in T, keep all ordinary marked points and the first extra marked point (if any) but remove all other extra marked points;

(2) Stabilize by removing any vertices that now have fewer than 3 special points and adjusting neighboring edges accordingly. (Note that since T is stable, this step can only affect vertices $\alpha \in T$ with $A_\alpha = 0$.)

By Lemma 2.2, any coherent almost complex structure $J \in \mathcal{J}_{\ell+1}$ determines for every nodal curve $\mathbf{z}$ modelled on T a $\Sigma_{\mathbf{z}}$-dependent almost complex structure $J_{\mathbf{z}}$ whose restriction to each component $S_\alpha \subset \Sigma_{\mathbf{z}}$ depends only on the special points on S_α. It follows that if $\mathbf{z}$ is modelled on T', then J uniquely determines a domain dependent almost complex structure on any component $S_\alpha \subset \Sigma_{\mathbf{z}}$ with $A_\alpha \neq 0$ (cf. the discussion preceding Corollary 5.9 in [CM07]). We can extend this to a $\Sigma_{\mathbf{z}}$-dependent almost complex structure

$$J_{\mathbf{z}} \in \mathcal{J}_{T'}$$

by setting $J_{\mathbf{z}}|_{S_\alpha}$ for each $\alpha \in T'$ with $A_\alpha = 0$ to match the fixed domain-independent reference almost complex structure J_1. In this way, we can speak of *nodal J-holomorphic maps* $(\mathbf{z}, \mathbf{u})$ modelled on the weighted $(m+\ell')$-labelled tree $(T', \{A_\alpha\})$; note that the definition of $J_{\mathbf{z}}$ on components S_α with $A_\alpha = 0$ plays no role here since $\mathbf{u}$ is necessarily constant on such components. Denote by $\widetilde{\mathcal{M}}_{T'}^{\{A_\alpha\}}(M, J; Z_D)$ the space of such maps for which the ℓ' extra marked points are all mapped into Z_D, and denote its quotient by the group of biholomorphic isomorphisms by

$$\mathcal{M}_{T'}^{\{A_\alpha\}}(M, J; Z_D) := \widetilde{\mathcal{M}}_{T'}^{\{A_\alpha\}}(M, J; Z_D)/ \sim .$$

There is a natural projection

$$\mathcal{M}_T^{\{A_\alpha\}}(M, J; Z_D) \to \mathcal{M}_{T'}^{\{A_\alpha\}}(M, J; Z_D)$$

defined by forgetting $\ell - \ell'$ of the extra marked points and then collapsing constant components as necessary in order to stabilize the domain. Since all the ordinary marked points are retained in this process, the factorization (A.1) of (ev, Φ) is well defined. The pseudocycle property now *mostly* follows from the following lemma, whose proof is exactly the same as [CM07, Lemma 5.6, Prop. 5.7 and Cor. 5.8].

Lemma A.2. *For generic J, if $e(T')$ denotes the number of edges in the tree T', then $\mathcal{M}_{T'}^{\{A_\alpha\}}(M, J; Z_D)$ is a smooth manifold with*

$$\dim \mathcal{M}_{T'}^{\{A_\alpha\}}(M, J; Z_D) = 2(n-3) + 2c_1(A) + 2m - 2e(T')$$
$$= \dim \mathcal{M}_{0,m+\ell}^A(M, J; Z_D) - 2e(T').$$

$\square$

We must still deal with the possibility that T has more than one vertex but T' has only one, in which case $\mathcal{M}_{T'}^{\{A_\alpha\}}(M, J; Z_D)$ can be regarded as a space of smooth (non-nodal) curves $\mathcal{M}_{0,m+\ell'}^A(M, J; Z_D)$ constrained to send their ℓ' extra marked points into Z_D.[3] This space has dimension equal to that of $\mathcal{M}_{0,m+\ell}^A(M, J; Z_D)$, but we claim that for generic J, if T has more than one vertex, then curves in $\mathcal{M}_{0,m+\ell'}^A(M, J; Z_D)$ that arise in this way from elements of $\mathcal{M}_T^{\{A_\alpha\}}(M, J; Z_D)$ lie in a subset of codimension at least 2. The crucial point here is that such a curve will never belong to the open subset

$$\mathcal{M}_{0,m+\ell'}^{A,*}(M, J; Z_D) \subset \mathcal{M}_{0,m+\ell'}^A(M, J; Z_D)$$

consisting of curves whose intersections with Z_D at the ℓ' extra marked points are all transverse, and for generic J, [CM07, §6] shows that the complement of this subset is a finite union of smooth submanifolds having dimension at most $\dim \mathcal{M}_{0,m+\ell'}^A(M, J; Z_D) - 2$. To see that curves in $\mathcal{M}_{0,m+\ell'}^{A,*}(M, J; Z_D)$ are excluded, observe that the curves in question arise precisely in situations where removing the relevant extra marked points from ghost bubbles in T makes all of them unstable—in particular, $(T, \{A_\alpha\})$ must in this case consist of the following:

- A unique vertex α_0 that has all m of the ordinary marked points and $A_{\alpha_0} = A \neq 0$;
- One or more ghost trees that each have no ordinary marked points but at least two of the extra marked points.

The resulting curve in $\mathcal{M}_{0,m+\ell'}^A(M, J; Z_D)$ is not contained in Z_D but has ℓ' marked points at which it must intersect Z_D, and if all of these ℓ' intersections are transverse, then the fact that $A \cdot [Z_D] = \ell > \ell'$ implies there must be additional intersections separate from the extra marked points. But since these curves are assumed to arise from objects in the closure of

---

[3]Since technically J belongs to $\mathcal{J}_{\ell+1}$ and not $\mathcal{J}_{\ell'+1}$, the definition of J-holomorphicity for curves in $\mathcal{M}_{0,m+\ell'}^A(M, J; Z_D)$ is a bit subtle and must be understood in the same sense as the preceding discussion of $\mathcal{M}_{T'}^{\{A_\alpha\}}(M, J; Z_D)$.

$\mathcal{M}_{0,m+\ell}^{A}(M, J; Z_D)$, the latter implies (via positivity of intersections) the existence of curves in $\mathcal{M}_{0,m+\ell}^{A}(M, J; Z_D)$ that have intersections with Z_D outside their extra marked points, and that is impossible. This proves:

Lemma A.3. *For generic J, if T has more than one vertex and T' has only one, then the restriction of (ev, Φ) to $\mathcal{M}_T^{\{A_\alpha\}}(M, J; Z_D)$ factors as*

$$\mathcal{M}_T^{\{A_\alpha\}}(M, J; Z_D) \to \mathcal{M}_{0,m+\ell'}^{A}(M, J; Z_D) \setminus \mathcal{M}_{0,m+\ell'}^{A,*}(M, J; Z_D)$$
$$\to M^m \times \overline{\mathcal{M}}_{0,m},$$

where the space in the middle is a finite union of manifolds having dimension at most $\dim \mathcal{M}_{0,m+\ell}^{A}(M, J; Z_D) - 2$. $\qquad\square$

It follows from Lemmas A.2 and A.3 that for generic J, (ev, Φ) is a pseudocycle as claimed. Using these same factorizations, one can similarly adapt the proof of [CM07, Theorem 1.3] to show that the rational pseudocycle defined by $\frac{1}{\ell!}(\mathrm{ev}, \Phi)$ is independent of the choices (J_0, Z_D, J) up to rational cobordism. This involves defining corresponding moduli spaces for 1-parameter families of data, as well as moduli spaces of curves with two sets of extra marked points constrained by two Donaldson hypersurfaces of differing degrees—the idea in each case is to factor (ev, Φ) as above through moduli spaces in which each ghost tree carries at most one extra marked point. Such moduli spaces always have small enough dimension to establish the pseudocycle condition.

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