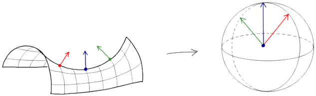
Semicontinuity of Gauss maps and the Schottky problem

Thomas Krämer, HU Berlin
(w/ Giulio Codogni, Roma II)



HU Algebraic Geometry Seminar, 11/11/2020

Setup:

- A abelian variety / $\mathbb{C}$
- $T^\vee A = A \times V$ its cotangent bundle
- $D \subset A$ a reduced divisor
- For each $p \in \mathrm{Sm}(D)$ we get a line

$$\gamma_D(p) \; := \; \ker\Big(V = T_p^\vee A \longrightarrow T_p^\vee D\Big) \; \subset \; V$$

- The Gauss map is the rational map

$$\gamma_D : \quad D \dashrightarrow \mathbb{P}V, \quad p \mapsto \gamma_D(p).$$

Remark. *For a reduced irreducible $D \subset A$ the following are equivalent:*

- *D is ample.*
- $\mathrm{Stab}(D) := \{x \in A \mid D + x = D\}$ *is finite.*
- $\gamma_D : D \dashrightarrow \mathbb{P}V$ *is a generically finite dominant cover.*

If these properties are not satisfied, then the divisor comes by pullback from an abelian quotient variety $A/\mathrm{Stab}(D)$ of smaller dimension and we can work directly there.

Q: What can we say about the generic degree $\deg(\gamma_D)$?

Example. *Let* (A, Θ) *be a ppav with* $\dim(A) = g$.

- *We have* $\deg(\gamma_\Theta) \leq g!$ *with equality iff* Θ *is smooth.*
- *If* $(A, \Theta) = \mathrm{Jac}(C)$ *is the Jacobian of a smooth projective curve, then*

$$\deg(\gamma_\Theta) = \begin{cases} 2^g & \text{if } C \text{ is hyperelliptic,} \\ \binom{2g-2}{g-1} & \text{otherwise.} \end{cases}$$

- *If* (A, Θ) *is a generic Prym variety, a result by Verra says that*

$$\deg(\gamma_\Theta) = D(g) + 2^{g-3} \quad \text{where} \quad D(g) = \cdots$$

The goal of today's talk is to explain the following answer to a conjecture by Codogni, Grushevsky and Sernesi:

Thm 1. *For any $d \in \mathbb{N}$ the Gauss loci*

$$\mathcal{G}_d \;=\; \{(A,\Theta) \in \mathcal{A}_g \mid \deg(\gamma_\Theta) \leq d\} \;\subseteq\; \mathcal{A}_g \quad \textit{are closed.}$$

Thm 2. *Inside $\mathcal{A}_g$ we have:*

- *The closure of the locus of Jacobians is a component of the Gauss locus $\mathcal{G}_d$ for $d = \binom{2g-2}{g-1}$.*
- *The closure of the locus of hyperelliptic Jacobians is a component of $\mathcal{G}_d$ für $d = 2^g$.*
- *The closure of the locus of Prym varieties is a component of $\mathcal{G}_d$ for $d = D(g) + 2^{g-3}$.*

More interestingly, our proof gives a new method to compute Gauss degrees by degeneration...

For theorem 2 we show that the stratification of $\mathcal{A}_g$ by the degree of the Gauss map refines the Andreotti-Mayer strata

$$\mathcal{N}_c \;=\; \{\, (A,\Theta) \in \mathcal{A}_g \mid \dim \mathrm{Sg}(\Theta) \geq c \,\} \quad \text{for } c \in \mathbb{N}.$$

More precisely:

Thm 3. *Let $\mathcal{N} \subset \mathcal{N}_c$ be an irreducible component whose general point corresponds to a ppav (A,Θ) where $\mathrm{Sg}(\Theta)$ has no negligible component. Then $\mathcal{N}$ is an irreducible component of the Gauss locus $\mathcal{G}_d$ for some $d \in \mathbb{N}$.*

Here a closed subvariety $Z \subseteq A$ is called negligible if comes by pullback from a proper abelian quotient variety, i.e. iff the stabilizer

$$\mathrm{Stab}(Z) := \{x \in A \mid Z + x = Z\} \quad \text{has positive dimension.}$$

Remark. *If A is simple as an abelian variety, it has no proper negligible closed subvarieties. Thus thm 3 $\Rightarrow$ thm 2.*

We don't know if the assumption in thm 3 is needed:

Q: *Are there indecomposable ppav's $(A, \Theta) \in \mathcal{A}_g$ such that the underlying reduced subscheme of the singular locus $\mathrm{Sg}(\Theta)$ has a negligible irreducible component?*

Our method also works for the intersection cohomology Euler characteristic $\chi_{\mathrm{IC}}(X)$ of subvarieties $X \subseteq A$. For instance:

Corollary 4. *For any $d \in \mathbb{N}$ the loci*

$$\mathcal{X}_d \;=\; \{(A,\Theta) \in \mathcal{A}_g \mid \chi_{\mathrm{IC}}(\Theta) \le d\} \;\subseteq\; \mathcal{A}_g \quad \textit{are closed.}$$

Corollary 5. *Inside $\mathcal{A}_g$, the closure of the locus of Jacobian varieties is an irreducible component of $\mathcal{X}_d$ for $d = \binom{2g-2}{g-1}$.*

Q: *The above answer to the Schottky problem only sees the homeomorphism type of the theta divisor. Does the same hold for all the loci $\mathcal{G}_d$ and $\mathcal{N}_c$?*

Ideas of the proof

- Gauss maps via conormal varieties
- Lagrangian specialization
- Kashiwara's index formula

Preliminary remark: Isn't semicontinuity obvious?

Example. Let $f : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ be the generically finite rational map defined by a linear system of four generic cubics passing through $n < 27$ given points.

Then

- $\deg(f) = 27 - n$ if the points are in general position.
- $\deg(f) = 20 - (n - \delta)$ if $\delta \geq 4$ of them are in general position on a line and the others in general position.

Moving the points, we can get families of rational maps where the degree jumps both *up* and *down* under specialization!

We have to show that for Gauss maps on abelian varieties such things do not happen:

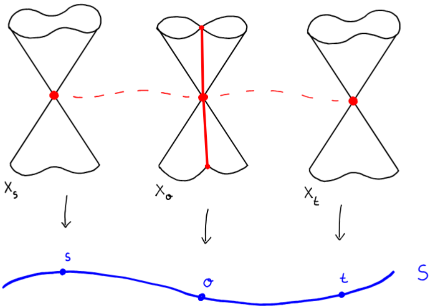
Semicontinuity theorem. *Let $A \to S$ be an abelian scheme and $D \subset A$ a relatively ample divisor which is flat over S, then for each $d \in \mathbb{N}$ the subsets*

$$S_d \;:=\; \{s \in S \mid \deg(\gamma_{D_s}) \leq d\} \;\subseteq\; S \quad \textit{are Zariski closed.}$$

Jump criterion. *Let $\dim S = 1$ and $0 \in S(\mathbb{C})$. If $\mathrm{Sg}(D_0)$ has a non-negligible component that is not in the closure of $\bigcup_{s \neq 0} \mathrm{Sg}(D_s)$, then*

$$\deg(\gamma_{D_0}) \;<\; \deg(\gamma_{D_s}) \quad \textit{for all nearby } s \neq 0.$$

E.g. in this picture the degree jumps down at $0 \in S$ unless the solid red line is negligible:



The proof of the semicontinuity theorem and jump criterion uses specialization of conormal varieties...

Conormal varieties

- Let W be an ambient smooth variety.
- For any closed subvariety $X \subseteq W$ consider the conormal variety

$$\begin{aligned} \Lambda_X &= \overline{\{(x,\xi) \mid x \in \mathrm{Sm}(X),\ \xi \in T_x^\vee W,\ \xi \perp T_x X\}} \\ &\subseteq T^\vee W \end{aligned}$$

- The conormal varieties are precisely the conic Lagrangian subvarieties of the cotangent bundle.
- We put $\mathcal{L}(W) = \bigoplus_{X \subseteq W} \mathbb{Z} \cdot \Lambda_X$.

Suppose W is projective of pure dimension n.

Definition. *The degree of a conic Lagrangian cycle is its intersection number with the zero section* $i : W \hookrightarrow T^\vee W$ *of the cotangent bundle:*

$$\deg : \quad \mathcal{L}(W) \longrightarrow \mathrm{CH}^n(T^\vee W) \xrightarrow{i^*} \mathrm{CH}^n(W) \twoheadrightarrow \mathbb{Z}$$

Remark. *The degree is the topological Euler characteristic weighted by the local Euler obstruction* $\mathrm{Eu}_X : W \to \mathbb{Z}$ *of MacPherson:*

$$\begin{aligned} \deg(\Lambda_X) &= (-1)^{\dim X} \cdot \chi_{\mathrm{top}}(X, \mathrm{Eu}_X) \\ &= (-1)^{\dim X} \cdot \chi_{\mathrm{top}}(X) \quad \text{if } X \text{ is smooth} \end{aligned}$$

The degree of conormal varieties can be negative. On abelian varieties this doesn't happen:

Positivity (Franecki-Kapranov, Weissauer). *If $W = A$ is an abelian variety, then*

- $\deg(\Lambda_X) \geq 0$ *for all* $X \subseteq A$.
- $\deg(\Lambda_X) = 0$ *iff* X *is negligible.*
- $\deg(\Lambda_X) = \deg(\gamma_X)$ *for the Gauss map*

$$\gamma_X : \quad \Lambda_X \subseteq T^\vee A = A \times V \xrightarrow{p_2} V$$

NB. If $X \subseteq A$ is a divisor, then after projectivizing the fibers of the cotangent bundle the above Gauss map resolves the classical Gauss map. So their degrees are the same.

To prove semicontinuity for Gauss maps of conormal varieties, we recast the definitions in a relative setting:

- Let $f : W \to S$ be a smooth proper morphism.
- For any closed subvariety $X \subseteq W$ consider the relative conormal variety

$$\begin{aligned} \Lambda_{X/S} &= \overline{\{(x,\xi) \mid x \in \mathrm{Sm}(X/S), \xi \perp T_x X_{f(x)}\}} \\ &\subseteq T^\vee(W/S) \end{aligned}$$

- As before we put

$$\mathcal{L}(W/S) = \bigoplus_{X \subseteq W} \mathbb{Z} \cdot \Lambda_{X/S}.$$

For $\dim(S) = 1$ we have:

Principle of Lagrangian specialization. *The map sending a relative conormal variety to its fiber over $s \in S(\mathbb{C})$ gives a homomorphism*

$$\mathrm{sp}_s : \quad \mathcal{L}(W/S) \longrightarrow \mathcal{L}(W_s), \quad \Lambda \mapsto \left[\Lambda \cdot f^{-1}(s)\right].$$

For any subvariety $X \subseteq W$ there exists a finite subset $\Sigma \subset S$ such that

$$\mathrm{sp}_s(\Lambda_{X/S}) = \begin{cases} \Lambda_{X_s} & \text{if } s \notin \Sigma, \\ m_s \Lambda_{X_s} + R_s & \text{if } s \in \Sigma, \end{cases}$$

w/ $m_s \geq 1$ and an effective cycle R_s supported over $\mathrm{Sg}(X_s)$.

Semicontinuity is now obvious:

Corollary. *Let $W = A \to S$ be an abelian scheme and $X \subseteq A$ a closed subvariety which is flat over S. Then the map*

$$S \ \to \ \mathbb{N}_0, \quad s \ \mapsto \ \deg(\Lambda_{X_s}) \quad \text{is lower semicontinuous.}$$

Proof.

- By flatness, $d := \deg(\mathrm{sp}_s(\Lambda_{X/S}))$ is independent of s.
- Write $\mathrm{sp}_s(\Lambda_{X/S}) = \Lambda_{X_s} + R_s$ for a cycle $R_s \geq 0$.
- Then we get

$$\deg(\Lambda_{X_s}) \ = \ d - \deg(R_s) \ \leq \ d$$

with $<$ at most for the finitely many s with $R_s \neq 0$. □

For the jump criterion we need to know if $R_s \neq 0$. This has nothing to do with abelian varieties, we work in the following setup:

- S is a smooth curve,
- $f : W \to S$ is a smooth dominant morphism,
- $X \subseteq W$ is a closed subvariety flat over S,
- $d := \operatorname{codim}(X, W)$.

Our jump criterion for divisors follows from:

Proposition. *Assume $d = 1$. If $\mathrm{Sg}(X/S)$ has an irreducible component Z which is contained in the fiber over $s \in S(\mathbb{C})$, then*

$$R_s \;:=\; \mathrm{sp}_s(\Lambda_{X/S}) \;-\; \Lambda_{X_s} \;\geq\; \Lambda_Z.$$

Sketch of proof. We may assume

- $Z = \mathrm{Sg}(X/S)$,
- $T^\vee(W/S) = W \times V$ is trivial,
- $X \subset W$ is cut out by a regular sequence $f_1, \dots, f_d$ (we don't assume $d = 1$ yet).

The relative Gauss map $X \dashrightarrow \mathrm{Gr}(d, V)$ is resolved by blowing up Z, and

$$\alpha_X : \quad \widehat{X} \;=\; \mathrm{Bl}_Z(X) \;\longrightarrow\; X \times \mathrm{Gr}(d, V)$$

is a closed immersion (use $\widehat{X} = \mathrm{Proj}_X \bigoplus_{n \geq 0} \mathcal{I}_Z^n$ & Plücker).

By base change to the flag variety we then get the following diagram:

$$\begin{array}{ccccc} Y & \hookrightarrow & X \times \mathrm{Fl}(d,1,V) & \longrightarrow & X \times \mathbb{P}V \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow & & \\ \widehat{X} & \xhookrightarrow{\alpha_X} & X \times \mathrm{Gr}(d,V) & & \end{array}$$

Let

- $\alpha_Y : Y \to X \times \mathbb{P}V$ be the composite of the top row,
- $E_Y := \pi^{-1}(E_X)$ for the exceptional divisor $E_X \subset \widehat{X}$.

Claim. For the support $\Lambda = \mathrm{Supp}(\mathrm{sp}_s(\Lambda_{X/S})) \subseteq X \times V$ we have

$$\alpha_Y(E_Y) \subseteq \mathbb{P}\Lambda.$$

This finishes the proof:

- We have $n := \dim(W_s) = \dim(E_Y)$.
- For $d = 1$ the morphism $\alpha_Y = \alpha_X$ is a closed immersion and we get
$$\dim(\alpha_Y(E_Y)) \;=\; n.$$
- Since Λ is of pure dimension n, it follows that
$$\alpha_Y(E_Y) \subseteq \Lambda \quad \text{is a union of components of } \Lambda.$$
- But $\Lambda \subseteq T^\vee W_s$ is conic Lagrangian and hence each of these components is a conormal variety.
- As $\alpha_Y(E_Y)$ surjects onto Z, we get $\Lambda_Z \subseteq \alpha_Y(E_Y)$. $\square$

Remark. *The same works also for* $d = \operatorname{codim}(X, W) > 1$ *if the morphism*

$$\alpha_Y : \quad E_Y \longrightarrow X \times \mathbb{P}V$$

is generically finite onto its image.

Q: *This seems a very mild condition. Does it always hold?*

$$\begin{array}{ccccccc} E_Y & \hookrightarrow & Y & \hookrightarrow & X \times \mathrm{Fl}(d,1,V) & \longrightarrow & X \times \mathbb{P}V \\ \downarrow & & \downarrow & & \downarrow & & \\ E_X & \hookrightarrow & \mathrm{Bl}_Z(X) & \underset{\text{Gauss}}{\hookrightarrow} & X \times \mathrm{Gr}(d,V) & & \end{array}$$

Final remark: Relation with topology

Setup:

- X complex algebraic variety
- $\mathrm{IH}^\bullet(X)$ intersection cohomology
- $\chi_{\mathrm{IC}}(X) := \sum_{i\in\mathbb{Z}}(-1)^{\dim X+i}\dim \mathrm{IH}^i(X)$.

Theorem. *Let $A \to S$ be an abelian scheme and $X \subseteq A$ a subvariety such that the map $X \to S$ is flat with generically reduced fibers. Then for all $d \in \mathbb{N}$ the subsets*

$$S_d := \{s \in S \mid \chi_{\mathrm{IC}}(X_s) \leq d\} \subseteq S \quad \textit{are Zariski closed.}$$

This is again a result about conormal degrees:

- Kashiwara's index formula says $\chi_{\mathrm{IC}}(X_s) = \deg(\mathrm{CC}(\delta_{X_s}))$ for the cycle

$$\mathrm{CC}(\delta_{X_s}) \;=\; \Lambda_{X_s} \;+\; \cdots \;\in\; \mathcal{L}(A_s).$$

- Specialization of conic Lagrangian cycles corresponds to the nearby cycles functor on perverse sheaves: A theorem by Ginzburg says

$$\mathrm{CC}(\Psi_s(P)) \;=\; \mathrm{sp}_s(\mathrm{CC}(P)) \quad \text{for all } P \in \mathrm{Perv}(A).$$

- Now apply this to perverse intersection complexes. □

Why perverse sheaves?

- For any ambient smooth projective variety W we have defined $\deg : \mathcal{L}(W) \to \mathbb{Z}$.
- For $X \subseteq W$ the degree

$$\deg(\Lambda_X) = (-1)^{\dim X} \cdot \chi_{\mathrm{top}}(X, \mathrm{Eu}_X)$$

  is in general not a topological invariant. A cuspidal cubic and a smooth rational curve have different Eu_X.
- But in the world of perverse sheaves we have

$$\deg\big(\mathrm{CC}(\delta_X)\big) \;=\; \deg\big(\Lambda_X + \cdots\big) \;=\; \chi_{\mathrm{IC}}(X)$$

  which only depends on the homeomorphism type of X.

Thank you very much!