Summer Term 2015: Perfectoid Algebras

The main local ingredients in Peter Scholze's groundbreaking concept of *perfectoid spaces* are the notion of *perfectoid algebras* and the tilting equivalence for perfectoid algebras. The aim of this seminar is trying to understand in some detail what perfectoid algebras are and why the tilting equivalence holds true.

Thus, we want to read chapters 3, 4 and 5 of the paper [Sch], focussing entirely on the *local* aspects of [Sch], i.e. on the underlying commutative algebra. For this, we will also need to consult additional literature pertaining to Faltings' idea of *almost mathematics* and the notion of cotangent complex.

Almost perfect(oi)d guides through the material of this seminar seem to be the short notes [B1] and [M], but one can easily find many more overviews in the www.

Throughout we should stick to the notations of [Sch] (i.e. even when reporting on [GR] where occasionally the notations deviate from those in [Sch]).

Talk 0 is independent of all the others (and could be omitted). Talk 12 can be omitted. Talks 7 and 8 might be contracted into a single talk.

0. An aperitif: A classical result of Tate

Let K be a finite field extension of $\mathbb{Q}_p$, let $\mathbb{C}_p$ be the completion of an algebraic closure $\overline{K}$ of K. Then $\mathbb{C}_p^{\mathcal{G}} = K$ for $\mathcal{G} = \mathrm{Gal}(\overline{K}/K)$. Tate's proof (of this result, and in fact of more) in section 3 of [T] inspired Falting's invention of almost mathematics.

1. Perfectoid Fields

[Sch] section 3. Give detailed proofs (in particular of [Sch] Lemma 3.4). Explain the example [Sch] p. 245 (cf. also [D] examples 4.1.8 and 4.1.9). The proof of [Sch] Theorem 3.7 will be given later.

2. Almost Mathematics: The basic notions

First part: [GR] 2.1.2, 2.1.3, 2.1.4. We always concentrate on the case where $\mathfrak{m}$ is *flat*, which often simplifies the exposition. (Cf. also [D] section 3.1.)

Second part: [Sch] 4.1 – 4.4, resp. [GR] 2.2.2. (Cf. also [D] section 3.2.)

3. The category K^{0a}-mod, the localization functor K^{0a}-mod$\to K^0$-mod and its right adjoint

First part: [Sch] 4.5, resp. [GR] 2.2.5, 2.2.6, 2.2.7. (Cf. also [D] section 3.3.)

Second part: [Sch] 4.6, resp. [GR] 2.2.9 – 2.2.14. (Cf. also [D] section 3.4 up to 3.4.18.)

4. Homological Algebra in K^{0a}-mod

[Sch] 4.7 – 4.11, resp. [GR] 2.2.16, 2.2.23. (Cf. also [D] section 3.4 from 3.4.19 onwards.)

5. Étale morphisms in almost mathematics

[Sch] 4.12 – 4.16, resp. [GR] 3.1.1 – 3.1.4, 4.1.14. (One may also briefly explain [GR] 3.4.4.)

6. The cotangent complex

[I] chapter II (section II.1 can be omitted or handled very briefly). We will not need the cotangent complex in the topos-theoretic generality as presented in [I] — for us it will be enough to introduce it as an object of commutative algebra. The cotangent complex (in an almost variation) will be needed in talk 7 as well as in talk 10.

7. Étale extensions along nilpotent thickenings I

[GR] 3.2.18 and all the preparations needed for the proof.

8. Étale extensions along nilpotent thickenings II

[Sch] 4.17, resp. [GR] 3.2.28, 5.3.27.

9. Extensions of perfectoid algebras from generic fibres to the almost integral level

[Sch] 5.1 up to the statement of Theorem 5.10.

10. Reducing almost integral perfectoid algebras. End of the proof of the tilting equivalence

[Sch] Proof of Theorem 5.10 (up to the top of page 279).

11. Falting's almost purity theorem in positive characteristic

[Sch] section 5, from top of page 279 onwards.

12. A light desert: Almost mathematics, Falting's purity theorem and a conjecture of Hoechster

[B].

References

[B] B. Bhatt: Almost direct summands, http://arxiv.org/pdf/1109.0356.pdf

[B1] B. Bhatt: Almost Ring Theory II: Perfectoid Rings, Notes taken by Dan Collins at the MSRI Hot Topics Workshop 'Perfectoid Spaces and their Applications'

[D] N. Diekert: Der Tilt von überkonvergenten Potenzreihen, Diplomarbeit, Universität Freiburg (2013), http://home.mathematik.uni-freiburg.de/arithmetische-geometrie/preprints/diekert-diplom.pdf

[GR] O. Gabber and L. Ramero: Almost ring theory. Lecture Notes in Mathematics **1800**, Springer Verlag, 2003

[I] L. Illusie: Complexe cotangent ét deformations I. Lecture Notes in Mathematics **239**, Springer Verlag, 1971

[M] Daniel Miller: Perfectoid rings, almost mathematics, and the cotangent complex, http://www.math.cornell.edu/ dkmiller/bin/almost-cotangent.pdf

[S] P. Scholze: Perfectoid Spaces. Publ. Math. Inst. Hautes Études Sci. **116**, 245 – 313 (2012)

[T] J. Tate: p-divisible groups. Proceedings of a Conference on Local Fields, Driebergen 1966, 158-183 (1967)