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Explicit Shintani base change and the Macdonald correspondence for characters of $\mathrm{GL}_n(k)$

Allan J. Silberger [a,*], Ernst-Wilhelm Zink [b]

[a] *Cleveland State University, 2121 Euclid Avenue, Cleveland, OH 44115, USA*
[b] *Humboldt-Universität, Institut für Mathematik, Rudower Chaussee 25, 10099 Berlin, Germany*

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Abstract

Let $\{K_d\}_{d\geqslant 1}$ denote the set of unramified extensions of a local field K and $\{k_d\}_{d\geqslant 1}$ the respective residual field extensions. The authors recall Macdonald's parameterization [I.G. Macdonald, Zeta functions attached to finite general linear groups, Math. Ann. 249 (1980) 1–15] of the irreducible characters of $\mathrm{GL}_n(k_d)$ in terms of "I-equivalence classes" of tame n-dimensional representations of the Weil–Deligne group $W'(K_d)$. Using Zelevinsky's PSH Hopf algebra theory [A. Zelevinsky, Representations of Finite Classical Groups, Lecture Notes in Math., vol. 869, Springer-Verlag, New York, 1981], they prove (see (1.1)) that $\mathcal{M}_n(k_d)\circ \mathrm{bc}_{k\uparrow k_d} = \mathrm{res}_{K\downarrow K_d}\circ \mathcal{M}_n(k)$, where $\mathcal{M}_n(k)$ denotes the Macdonald parameterization map for $\mathrm{GL}_n(k)$, $\mathrm{bc}_{k\uparrow k_d}$ the Shintani base-change map for GL_n, and $\mathrm{res}_{K\downarrow K_d}$ the restriction of n-dimensional representations from the Weil–Deligne group $W'(K)$ to $W'(K_d)$ for I-equivalence classes of tame representations. As Henniart [G. Henniart, Sur la conjecture de Langlands locale pour GL_n, J. Théor. Nombres Bordeaux 13 (2001) 167–187] has shown, the same relation holds with $\mathcal{M}_n$ replaced by the local Langlands correspondence and finite-field base change replaced by local-field base change with no restriction to I-equivalence classes. In an Addendum the authors show (see (A.1)) that the map φ_0 which sends a level-zero irreducible representation of $\mathrm{GL}_n(K)$ to the reduction of its "tempered type" [P. Schneider, E.-W. Zink, K-types for the tempered components of a p-adic general linear group, J. Reine Angew. Math. 517 (1999) 161–208] connects the level-zero local-field Langlands parameterization to the finite-field parameterization of Macdonald. They also remark (see the concluding *Remark*) that φ_0 is compatible with the Shintani and local-field base change maps.

* Corresponding author.
E-mail addresses: a.silberger@csuohio.edu (A.J. Silberger), zink@mathematik.hu-berlin.de (E.-W. Zink).

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0. Introduction

In previous work [SZ2] the authors discovered that the "Abstract Matching Theorem" (AMT) (see [DKV] and [Ba]) implies explicit Shintani base change for all irreducible cuspidal characters of general linear groups over finite fields (see [SZ2, 5.7] and 9.1 below for a restatement). In this paper the authors answer affirmatively the natural follow-up question: What about the general case? Given a knowledge of explicit base change for irreducible cuspidal characters of $\mathrm{GL}_n(k)$ (k a finite field), is it possible to determine the images under base change of arbitrary irreducible characters of $\mathrm{GL}_n(k)$?

In order to formulate a general statement concerning explicit base change for general linear groups one must have a parameterization of the irreducible characters of $\mathrm{GL}_n(k)$. The most natural and general such parameterization occurs in the paper of I.G. Macdonald [Mac1], in which, by modifying the local Langlands correspondence between the set of admissible n-dimensional representations of the Weil–Deligne group of the local field K and the set of irreducible characters of $\mathrm{GL}_n(K)$, Macdonald constructed a correspondence which applies to $\mathrm{GL}_n(k)$, k the finite residual field of K. A few years later A. Fröhlich (see the remark between [Fr, (8.29)b&c]) hypothesized the existence of a base change theory for finite GL_n which preserves Macdonald's correspondence under restriction of representations of the Weil–Deligne group (also see [Fr, (7.1) and the list of properties on p. 52]). In this paper the authors prove that, with base change for finite GL_n defined as the inverse of the descent mapping defined by Shintani [Sh], the properties hypothesized by Fröhlich hold. In particular, the diagram (1.1) below is commutative.

In addition to assuming Macdonald's parameterization of the irreducible characters of $\mathrm{GL}_n(k)$, the present paper depends upon Shintani's fundamental theorem [Sh, Theorem 1], which asserts that *Shintani descent* induces a bijection from the set of Galois invariant irreducible characters of $\mathrm{GL}_n(k_d)$ to the set of all irreducible characters of $\mathrm{GL}_n(k)$ ($k_d|k$ an extension of degree d). When the authors speak of *base change*, or more properly *Shintani base change*, they mean the inverse of Shintani's descent mapping (see [Sh, Theorem 1] and [SZ2, §B], which gives a general introduction to Shintani descent theory). By "explicit" base change or descent the authors mean the description of these mappings in terms of Macdonald's parameterizations of the sets of irreducible characters for the respective groups.

As mentioned above, the authors assume [SZ2, 5.7], which gives the explicit descent of all irreducible characters of $\mathrm{GL}_n(k_d)$ which have cuspidal descent. An essential building block for the work presented here, this result seems to be outside the reach of the present paper's methods. [SZ2, 5.7] is not a new theorem; Gyoja [Gy] and Digne [Di] have proved more general statements regarding Deligne–Lusztig induction, and, moreover, their work already implies the transitivity of the descent mapping in the tower of characters lying over any cuspidal character of $\mathrm{GL}_n(k)$. Apparently no one has studied, at least in the context of finite fields, the functoriality of base change for unipotent characters and their generalizations, so the authors' extension of the results of Gyoja and Digne to this case (for general linear groups) may be new. In summary, the authors determine explicitly the Shintani base change mapping for $\mathrm{GL}_n(k)$, and they prove that this base change mapping makes the diagram (1.1) commutative. Put another way, they describe the Shintani base change mapping in terms of Macdonald's parameterizations and they prove the transitivity of Shintani base change and descent by proving that Shintani base change is compatible, via the Macdonald correspondence, with the restriction of tame n-dimensional representations of Weil–Deligne groups.

The main tool which the authors apply in the main part of this paper is Zelevinsky's powerful and beautiful PSH Hopf algebra theory for finite general linear groups. Explicit base change for unipotent characters, actually for all characters of finite general linear groups, occurs naturally as a consequence of the rigidity of the PSH algebra structures which encompass the character theories of symmetric groups and especially the theory of primary characters of finite general linear groups. PSH algebras were defined and studied by Zelevinsky [Zel1] and Springer/Zelevinsky [SpZ] (see also [Mac2]). The present paper develops the slight generalization of PSH algebra theory which applies to the present context (cf. Section 7).

We have added an Addendum (Appendix A) which proves the commutativity of another square diagram (see (A.1)). Put into words, this diagram asserts that in the level-zero case the (vertical) mapping φ_0, which sends an irreducible level-zero representation of $\mathrm{GL}_n(K)$ to the reduction of its tempered type, connects the diagram in which the two horizontal sides are, respectively, the level-zero local Langlands correspondence and the Macdonald parameterization map and the remaining vertical side is the map which assigns a tame n-dimensional representation of the Weil–Deligne group to its I-equivalence class. In concluding Appendix A we also remark that the mappings φ_0 connect the finite field and local field level-zero base-change maps.

1. The Macdonald correspondence and transitivity

Let $k := \mathbb{F}_q$ be a finite field, set $G_n = G_n(k) := \mathrm{GL}_n(k)$ for all $n > 0$ and set $G_0 := \{1\}$. For $n \geqslant 0$ let $\Omega_n := \Omega_n(k)$ denote the set of irreducible characters of G_n and set $\Omega := \coprod_{n\geqslant 0} \Omega_n$, the disjoint union of the sets of irreducible characters.

Let K denote a local field with k as its residual field and let $W'_K = \mathbb{C} \rtimes W_K$ be the Weil–Deligne group of K. In W_K we consider the inertial subgroup I and the wild ramification subgroup P. These groups are normal in W'_K, since the action of W_K on $\mathbb{C}$ factors through W_K/I. Let $\mathrm{Rep}_n(W'_K/P)_I$ be the set of all n-dimensional admissible representations of W'_K/P modulo a certain equivalence relation introduced by Macdonald [Mac1, (4.1) & (4.2)]. As an off-shoot of the level zero local Langlands correspondence, Macdonald constructed a bijective correspondence between Ω_n and $\mathrm{Rep}_n(W'_K/P)_I$ which preserves L and ϵ-factors.

Let $K_d|K$ be an unramified extension of degree $d \geqslant 1$; write $k_d|k$ for the extension of residual fields. In the diagram (1.1) below the vertical arrows, from left to right, denote Shintani base change and the restriction mapping for admissible representations of Weil–Deligne groups. The horizontal arrows, $\mathcal{M}_n(k)$ and $\mathcal{M}_n(k_d)$, denote Macdonald correspondences for $\mathrm{GL}_n(k)$ and $\mathrm{GL}_n(k_d)$, respectively.

In this paper, by introducing and analyzing a "twisted" model of Zelevinsky's finite field Hopf algebra [Zel1,SpZ], we are going to prove that the diagram (1.1) is commutative. Since the restriction mappings are obviously transitive, the commutativity of (1.1) implies that the Shintani base change maps are also transitive.

$$\begin{array}{ccc} \Omega_n & \xrightarrow{\mathcal{M}_n(k)} & \mathrm{Rep}_n(W'_K/P)_I \\ \Big\downarrow{\scriptstyle \mathrm{bc}} & & \Big\downarrow{\scriptstyle \mathrm{res}} \\ \Omega_n(k_d) & \xrightarrow{\mathcal{M}_n(k_d)} & \mathrm{Rep}_n(W'_{K_d}/P)_I \end{array} \tag{1.1}$$

The commutativity of (1.1) also implies that the Shintani base change mapping $\mathrm{bc}_{k\uparrow k_d} : \Omega_n \xrightarrow{\sim} \Omega_n(k_d)^\phi$ from the set of irreducible characters of G_n to the set of Galois-invariant, irreducible characters of $G_n(k_d)$ has an explicit description in terms of the Macdonald parameterizations of Ω_n and $\Omega_n(k_d)$ (cf. (5.1) and 9.6).

2. Zelevinsky's PSH algebra

We want to apply Zelevinsky's theory of PSH algebras, so we shall review some of the salient facts concerning the structure theory of these algebras. For details the reader will have to consult Zelevinsky's book [Zel1], where the theory is developed ab initio.

For $n \geqslant 0$ let $R_n := R_n(k)$ denote the free $\mathbb{Z}$-module spanned by the canonical basis Ω_n and let $R := \bigoplus_{n\geqslant 0} R_n$ be the graded $\mathbb{Z}$-module with the basis Ω. Thus R_n is the $\mathbb{Z}$-module of all virtual characters of G_n for all $n > 0$ and $R_0 = \mathbb{Z}$. Zelevinsky defines an inner product on R by making Ω an orthonormal basis for R. He defines the graded multiplication $m_{r,s} : R_r \otimes R_s \longrightarrow R_{r+s}$ by setting, for any basis elements $\rho \in \Omega_r$ and $\sigma \in \Omega_s$,

$$m_{r,s}(\rho, \sigma) := I(G_r \times G_s, G_{r+s}, \rho \otimes \sigma),$$

where the right side denotes the induction of the inflation to $P(k)$ of $\rho \otimes \sigma$ from any parabolic subgroup $P(k) = (G_r(k) \times G_s(k)) \ltimes U_P(k)$ to $G_{r+s}(k)$. And he defines the graded comultiplication

$$m^{*,n} : R_n \longrightarrow \bigoplus_{\{r,s\geqslant 0: r+s=n\}} R_r \otimes R_s$$

such that, for every $\tau \in \Omega_n$, $m^{*,n}_{r,s}(\tau)$ is the Jacquet/Harish-Chandra restriction of τ to the Levi subgroup $G_r(k) \times G_s(k) \subset G_n(k)$; this mapping is well defined because it is independent of the choice of a k-parabolic subgroup $P = (G_r \times G_s) \ltimes U_P$:

$$m^{*,n}_{r,s}(\tau)(x) := \frac{1}{|U_P(k)|} \sum_{u \in U_P(k)} \tau(xu) \quad \bigl(x \in G_r(k) \times G_s(k)\bigr).$$

With the multiplication $m := \bigoplus m_{r,s}$ and the comultiplication $m^* := \bigoplus m^{*,n}$ the tuple (R, m, m^*, Ω) is a graded, commutative and cocommutative Hopf algebra which is "positive" and "self-adjoint." The positive elements of the algebra are the elements which may be represented as integer linear combinations of the elements of Ω with nonnegative coefficients. Zelevinsky calls the resulting structure a PSH algebra (cf. [SpZ, §1]; also see [Zel1, 1.4] for the definition of "self-adjoint").

Let $\mathcal{C}_n \subset \Omega_n$ denote the set of all irreducible cuspidal characters of G_n ($n > 0$) and set $\mathcal{C} := \coprod_{n>0} \mathcal{C}_n$. In Zelevinsky's theory the elements of $\mathcal{C}$ are the *primitive irreducible elements* of R. For any $\pi \in \mathcal{C}$ let $\Omega(\pi) \subset \Omega$ be the subset consisting of all irreducible characters with cuspidal support a multiple of π and let $R(\pi)$ be the submodule of R spanned by $\Omega(\pi)$. Then, since m and m^* stabilize $R(\pi)$ and its tensor powers, $R(\pi)$ is a subalgebra of R, called the *primary* PSH-subalgebra of R with π as its unique primitive irreducible element. Zelevinsky [Zel1, §2] proves that $R \cong \bigotimes_{\pi\in\mathcal{C}} R(\pi)$, the limit of finite tensor products of primary PSH subalgebras.[1]

[1] Primary PSH-subalgebras are called "universal PSH-algebras" in [Zel1].

Zelevinsky also develops the structure theory of primary PSH-algebras. He proves that all primary PSH-algebras are isomorphic as PSH-algebras; concretely, every primary PSH-algebra is isomorphic to the PSH-algebra built from the set of characters of the finite symmetric groups S_n: Let $R(S) := \bigoplus_{n\geqslant 0} R(S_n)$ be the free $\mathbb{Z}$-module with the canonical basis $\Omega(S) := \coprod_{n\geqslant 0} \Omega(S_n)$, where for all $n \geqslant 1$ the basis set $\Omega(S_n)$ is the set of irreducible characters of S_n ($\Omega(S_0) = \{1\}$ and $R(S_0) = \mathbb{Z}$). The unique "primitive irreducible element" in $R(S)$ is the trivial representation $\mathbf{1} \in \Omega(S_1)$. The Hopf multiplication in $R(S)$ is generated by setting

$$m_{r,s}(\rho,\sigma) := \rho\cdot\sigma = \mathrm{ind}_{S_r\times S_s}^{S_{r+s}}(\rho\otimes\sigma) \quad \big(\rho\in\Omega(S_r),\ \sigma\in\Omega(S_s)\big)$$

and the comultiplication is generated by

$$m_{r,s}^{*,n}(\tau) := \mathrm{res}_{S_n}^{S_r\times S_s}(\tau) \quad \big(\tau\in\Omega(S_n)\big); \qquad m^{*,n} := \bigoplus_{(r,s):r+s=n} m_{r,s}^{*,n}.$$

Of fundamental importance is the fact that, for any primary PSH-algebra, there is exactly one nontrivial automorphism of the graded Hopf algebra structure.[2] To define this automorphism for the symmetric group algebra we first represent $R(S)$ in two ways as a (free) polynomial algebra over $\mathbb{Z}$ in infinitely many variables:

$$R(S) \cong \mathbb{Z}[x_1, x_2, x_3, \ldots] = \mathbb{Z}[y_1, y_2, y_3, \ldots].$$

We choose the variables $x_0 = y_0 = 1$, $x_1 = y_1 = \mathbf{1} \in \Omega(S_1)$, $x_i = \mathrm{sign} \in \Omega(S_i)$ and $y_i = \mathbf{1} \in \Omega(S_i)$ for $i \geqslant 2$. It follows that the respective powers of the variables are graded such that

$$\deg\big(x_i^k\big) = \deg\big(y_i^k\big) = ki \quad (k, i \geqslant 0);$$

moreover the variables satisfy the Hopf product relations

$$\sum_{i=0}^{n} (-1)^i x_i y_{n-i} = 0 \quad (n \geqslant 1).$$

The nontrivial automorphism of $R(S)$ switches x_n and y_n for all n.

To fix an isomorphism $R(S) \xrightarrow{\sim} R(\pi)$ we set $x_1(\pi) = y_1(\pi) = \pi$ and for each $i \geqslant 2$ we map $x_i \longmapsto x_i(\pi)$, the generalized Steinberg (GS) component, and $y_i \longmapsto y_i(\pi)$, the generalized trivial (GT) component of the Hopf power π^i. We recall that the GS and GT components of π^i are the only components which occur with multiplicity one, and, similarly, the trivial and sign characters are the only components of the regular representation of S_i which occur simply, so there are only two possible identifications of the x_i and y_i with irreducible characters in $\Omega_i(\pi)$ which preserve inner products. Note that we have chosen $x_i(\pi)$ to be the unique component of π^i which has a Whittaker vector.

For $\pi \in \mathcal{C}_s$, we set $d(\pi) := s$. Let $\Omega(\pi)$ denote the set of all elements of Ω which have cuspidal support a multiple of π. The isomorphism $R(S) \cong R(\pi)$ induces a canonical bijection between $\Omega(S)$ and $\Omega(\pi)$. Under this bijection $\Omega(S_n)$ corresponds to $\Omega_{nd(\pi)}(\pi)$, since every

[2] The automorphism preserves positivity and self-adjointness.

primary irreducible character with *cuspidal support* (see below, prior to 2.2) a multiple of π must have degree a multiple of $d(\pi)$. Thus $\Omega(\pi) = \coprod_{n\geqslant 0} \Omega_{nd(\pi)}(\pi)$.

For $\lambda := (\ell_1 \geqslant \cdots \geqslant \ell_r > 0)$, a nonzero partition of weight $|\lambda| = \ell_1 + \cdots + \ell_r$, define the Hopf products

$$x_\lambda := x_{\ell_1} \cdots x_{\ell_r} \in R(S_{|\lambda|}) \quad \text{and} \quad y_\lambda := y_{\ell_1} \cdots y_{\ell_r} \in R(S_{|\lambda|})$$

and, for $\pi \in \mathcal{C}$, the Hopf products

$$x_\lambda(\pi) := x_{\ell_1}(\pi) \cdots x_{\ell_r}(\pi) \in R_{|\lambda| d(\pi)} \quad \text{and} \quad y_\lambda(\pi) := y_{\ell_1}(\pi) \cdots y_{\ell_r}(\pi) \in R_{|\lambda| d(\pi)}.$$

Let $\lambda^t = 1^{\ell_1-\ell_2}, 2^{\ell_2-\ell_3}, 3^{\ell_3-\ell_4}, \ldots$ denote the transpose of the partition λ, i.e. the partition corresponding to the transpose of the Ferrer's diagram of λ. In the symmetric group case, there is a unique irreducible character of $S_{|\lambda|}$ which occurs as a common component of both x_λ and y_{λ^t}; we write $\{\lambda\} := x_\lambda \cap y_{\lambda^t}$ for this irreducible character, which occurs with multiplicity one as a component of x_λ and y_{λ^t}. Thus,

$$\langle \{\lambda\}, x_\lambda \rangle_{S_{|\lambda|}} = \langle \{\lambda\}, y_{\lambda^t} \rangle_{S_{|\lambda|}} = 1$$

with respect to the normalized measure on $S_{|\lambda|}$. Similarly (cf. [Zel1, 4.1]), there is a unique irreducible component common to both $x_\lambda(\pi)$ and $y_{\lambda^t}(\pi)$; we write $\pi^\lambda := x_\lambda(\pi) \cap y_{\lambda^t}(\pi)$ to denote this component, which occurs simply in both $x_\lambda(\pi)$ and $y_{\lambda^t}(\pi)$. Thus, with respect to the normalized measure on $G_{|\lambda| d(\pi)}$,

$$\langle \pi^\lambda, x_\lambda(\pi) \rangle_{G_{|\lambda| d(\pi)}} = \langle \pi^\lambda, y_{\lambda^t}(\pi) \rangle_{G_{|\lambda| d(\pi)}} = 1.$$

From the definition we see, in particular, that, for $\lambda = (|\lambda|)$, $\pi^\lambda = x_{|\lambda|}(\pi)$ and $\pi^{\lambda^t} = \pi^{1^{|\lambda|}} = \pi^{(1,1,\ldots,1)} = y_{|\lambda|}(\pi)$.[3]

2.1. Proposition.

(i) *For $\pi \in \mathcal{C}$ the mapping $\lambda \longmapsto \pi^\lambda$ defines a bijection between the set $\Lambda - \{(0)\}$[4] of all nonzero partitions and the set $\Omega(\pi)$ of all elements of Ω which have cuspidal support a multiple of π.*

(ii) *More precisely, $\{\lambda\} \in \Omega(S_n)$ if and only if $|\lambda| = n$ and $\pi^\lambda \in \Omega_{nd(\pi)}(\pi)$ if and only if $|\lambda| = n$. In the symmetric group algebra $R(S)$ the Hopf product $\mathbf{1}^n$ is the regular representation of S_n and $\mathbf{1}^n = \bigoplus_{|\lambda|=n} m_\lambda \{\lambda\}$, where m_λ denotes both the degree (dimension) and the multiplicity of $\{\lambda\}$ in $\mathbf{1}^n$. In the algebra $R(\pi)$ the Hopf product $\pi^n = \bigoplus_{|\lambda|=n} m_\lambda \pi^\lambda$; in this case m_λ is the multiplicity of π^λ but it is not the degree of π^λ.*

For $\pi \in \mathcal{C}$ we call $\Omega(\pi) \subset \Omega$ the set of irreducible π-*primary* characters.

[3] The symbols "x" and "y" are switched relative to the usage of [Zel1] in both the symmetric group and the general linear group cases.

[4] The partition (0) of 0, which has no positive parts, corresponds to the unique representation $\{(0)\}$ of S_0 and to the representation $\pi^{(0)} = 1$ of G_0.

Let $\mathrm{Div}^+(\mathcal{C})$ denote the semigroup of effective divisors over $\mathcal{C}$. For a divisor $D = \sum_{\pi\in\mathcal{C}} m_\pi \pi \in \mathrm{Div}^+(\mathcal{C})$ define

$$\deg(D) := \sum_{\pi\in\mathcal{C}} m_\pi d(\pi).$$

If $\tau \in \Omega_n$, then for a unique divisor $D(\tau) = \sum_{\pi\in\mathcal{C}} m_\pi(\tau)\pi$ such that $\deg(D(\tau)) = n$ the Hopf product satisfies

$$\left\langle \tau, \prod_{\pi\in\mathcal{C}} \pi^{m_\pi(\tau)} \right\rangle_{G_n} \neq 0.$$

We call $D(\tau)$ the *cuspidal support* of τ. We reformulate the following classical result:

2.2. Proposition. *Let $\tau \in \Omega_n$ and let $D(\tau) = \sum_{\pi\in\mathcal{C}} m_\pi(\tau)\pi$. Then τ has a unique Hopf product factorization of the form*

$$\tau = \prod_{\pi\in\mathcal{C}} \tau(\pi), \tag{2.1}$$

where $\tau(\pi) \in \Omega(\pi)$ and $D(\tau(\pi)) = m_\pi(\tau)\pi$. In other words, $\tau(\pi) = \pi^{\lambda_\pi}$ for some partition λ_π of $m_\pi(\tau)$ ($|\lambda_\pi| = m_\pi(\tau)$). Moreover, if τ and τ' are irreducible characters with disjoint cuspidal support, then the Hopf product $\tau\tau'$ is irreducible.

Proof. As noted above, $R \cong \bigotimes_{\pi\in\mathcal{C}} R(\pi)$, the right side being understood as the direct limit over tensor products containing only finitely many factors. The map from right to left is the Hopf product in R and the basis set of irreducible elements $\Omega = \coprod_{n\geqslant 0} \Omega_n$ satisfies $\Omega_n = \coprod_{n_1 d(\pi_1)+\cdots+n_r d(\pi_r)=n} \Omega_{n_1}(\pi_1) \times \cdots \times \Omega_{n_r}(\pi_r)$. □

We have noted Zelevinsky's structural results to the effect that for any $\pi \in \mathcal{C}$ the primary PSH algebra $R(\pi) \cong \mathbb{Z}[x_1, x_2, x_3, \ldots]$. Moreover, since every element of R belongs to a finite tensor product of primary PSH subalgebras and since the Hopf product is commutative, the algebra R is also isomorphic to a free polynomial ring over $\mathbb{Z}$ in countably many variables. Therefore, since a free polynomial ring over a UFD is a UFD, R is a UFD.

3. The set $P(\mathcal{X})^\Gamma$ as a parameter set for Ω

For $\bar{k}$ an algebraic closure of k consider the set of all extension fields $k \subseteq k_d \subset \bar{k}$ ($d \geqslant 1$), partially ordered by inclusion, and the direct limit group of the character groups $X(k_d^\times)$ with respect to the conorm mappings N^*:

$$\mathcal{X} := \varinjlim_{N^*} X\left(k_d^\times\right) = X\left(\varprojlim_{N} k_d^\times\right).$$

Let Λ denote the set of all partitions and consider the set

$$P(\mathcal{X}) := \{\lambda : \mathcal{X} \ni \chi \longmapsto \lambda_\chi \in \Lambda\}$$

of partition-valued functions λ on $\mathcal{X}$ such that $\lambda_\chi = (0)$ except for finitely many $\chi \in \mathcal{X}$.

The group $\Gamma = \Gamma_k := \mathrm{Gal}(\bar{k}|k)$ acts naturally on $\mathcal{X}$ and on $P(\mathcal{X})$. Let $P(\mathcal{X})^\Gamma$ denote the subset consisting of all Γ-invariant functions in $P(\mathcal{X})$. Since $\lambda \in P(\mathcal{X})^\Gamma$ is constant on any Galois orbit $[\chi] = \Gamma\chi$ $(\chi \in \mathcal{X})$, we may regard the elements of $P(\mathcal{X})^\Gamma$ as partition-valued functions defined on the set of Galois orbits $\Gamma\backslash\mathcal{X}$.

We define a bijective mapping $P(\mathcal{X})^\Gamma \longrightarrow \Omega(k)$ as follows:

$$P(\mathcal{X})^\Gamma \ni \lambda \longmapsto \tau_\lambda := \prod_{[\chi]\in\Gamma\backslash\mathcal{X}} \pi(\chi)^{\lambda_\chi} \in \Omega(k), \tag{3.1}$$

where $\pi(\chi)$ denotes the irreducible cuspidal character with the Green's parameter $[\chi]$. If the orbit $[\chi]$ has order s, then χ may be regarded as a regular character in $X(k_s^\times)$ (see 3.1(ii) below) and therefore $\pi(\chi) \in \mathcal{C}_s$.

3.1. Remarks.

(i) The character τ_λ is primary if and only if λ is supported on a single Γ orbit $[\chi]$, and it is cuspidal if and only if, in addition, $\lambda_\chi = (1)$.

(ii) Let $\chi \in \mathcal{X}$. Then the Γ orbit $[\chi]$ is finite, so the action of Γ on $[\chi]$ factors through $\mathrm{Gal}(k_s|k)$ for some smallest integer $s \geqslant 1$. Therefore, we may identify $[\chi]$ with an orbit of k-regular characters $[\chi_s] \subset X(k_s^\times)$. Thus, $[\chi_s]$ is seen to be the Green's parameter of $\pi(\chi_s) \in \mathcal{C}_s$ in the usual sense.

(iii) The group $\mathcal{X}$ does not depend upon the ground field k. Nevertheless, given $\chi \in \mathcal{X}$ the cuspidal characters $\pi(\chi) := \pi_k(\chi)$ and $\pi_{k_d}(\chi)$, where $k_d|k$ is a finite extension, are certainly different; $\pi(\chi) \in G_n^\wedge$, where n is the length of the Γ-orbit of χ, and $\pi_{k_d}(\chi) \in G_{n/(d,n)}(k_d)^\wedge$, $n/(d,n)$ being the length of the Γ_{k_d}-orbit of χ. For details concerning the relationship between these two cuspidal characters the reader may consult [SZ1, Proposition 1.1].

4. Macdonald's correspondence $\mathcal{M}(k): \Omega \longrightarrow P(\mathcal{X})^\Gamma \longrightarrow \mathrm{Rep}(W_K'/P)_I$

Recall that K is a local field with the finite field k as its residual field.

In order to use the bijection (3.1) to define the Macdonald correspondence we return to the Weil–Deligne group $W_K' = \mathbb{C} \rtimes W_K$, with I and P the inertial and wild ramification groups as in Section 1. On W_K we have the norm map $w \longmapsto \|w\| \in (\mathbb{R}_+)^\times$, which is defined such that $\bar{w}(x) = x^{\|w\|}$ for all $x \in \bar{k}$, where $\bar{w} \in \Gamma$ is induced by $w \in W_K$. The multiplication in W_K' is:

$$(z_1, w_1) \cdot (z_2, w_2) = \big(z_1 + \|w_1\| z_2, w_1 w_2\big).$$

Every admissible representation of W_K' may be regarded as a pair (N, ρ), where:

(i) ρ is a finite-dimensional $\mathbb{C}$-representation of W_K such that $\rho(w)$ is semisimple for all $w \in W_K$ and $\mathrm{Ker}(\rho)$ contains an open subgroup of the inertia subgroup I.

(ii) N is a nilpotent endomorphism of the representation space V_ρ such that

$$\rho(w) N \rho(w)^{-1} = \|w\| N \quad (w \in W_K).$$

This implies that from the pair (N, ρ) we may construct the representation

$$W_K' \ni (z, w) \longmapsto \exp(zN)\rho(w) \in \mathrm{Aut}(V_\rho).$$

Two admissible representations (N,ρ) and (N',ρ') are called I-equivalent (see [Mac1, §3]), if their restrictions to the direct product $\mathbb{C}\times I\subset W'_K$ are equivalent, i.e. if there exists a linear isomorphism $\alpha: V_\rho \longrightarrow V_{\rho'}$ such that $(N',\rho'|_I)=\alpha(N,\rho|_I)\alpha^{-1}$.

We write $\mathrm{Rep}(W'_K)$ (respectively, $\mathrm{Rep}(W'_K)_I$) for the set of equivalence classes (respectively, for the set of I-equivalence classes) of representations of W'_K. In this paper we consider only representations of the factor group W'_K/P.

For all $m\geqslant 1$ we define the *special representation* of dimension m:

$$\mathrm{sp}_K(m) := (N_m,\rho_m),$$

where N_m denotes the nilpotent $m\times m$ matrix with 1s on the sub-diagonal and elsewhere 0s, and

$$\rho_m(w)=\mathrm{diag}\big(\|w\|^{(1-m)/2},\|w\|^{(3-m)/2},\ldots,\|w\|^{(m-3)/2},\|w\|^{(m-1)/2}\big).$$

Clearly, $\rho_m(w)N_m\rho_m(w)^{-1}=\|w\|N_m$. Note that $\mathrm{sp}_K(1)=(0,1)$ is the trivial representation, and for all $m\geqslant 1$

$$\det\big(\exp(zN_m)\rho_m(w)\big)=\det\big(\rho_m(w)\big)=1 \quad (z\in\mathbb{C},\ w\in W_K).$$

It is well known that all irreducible representations of W'_K have the form $(0,\rho)$, where ρ is an irreducible representation of W_K and $N=0$, whereas all indecomposable representations of W'_K are of the form

$$\mathrm{sp}_K(m)\otimes\rho=(N_m,\rho_m)\otimes(0,\rho)=\big(\mathrm{diag}(N_m,\ldots,N_m),\rho_m\otimes\rho\big),$$

the multiplicity of N_m being $\dim(\rho)$, since

$$\exp(N_m)\otimes\exp(0_d)=\exp(N_m)\otimes I_d.$$

Obviously, $\det(\rho_m\otimes\rho)=\det(\rho)^m$. If $\lambda=(\ell_1\geqslant\ell_2\geqslant\cdots\geqslant\ell_r>0)$ is a nonzero partition, then we set

$$\mathrm{sp}_K(\lambda):=\mathrm{sp}_K(\ell_1)\oplus\cdots\oplus\mathrm{sp}_K(\ell_r),$$

a direct sum of special representations.

Next we use class field theory to identify the group of characters $\mathcal{X}$ with the group of characters of a subquotient of the Weil group. For this we consider the subquotient I/P of Γ_K. Since I/P is abelian (in fact, procyclic), class field theory implies that I/P is isomorphic to the projective limit $\varprojlim_N U_{K_d}/U^1_{K_d}$ of unit mod principal unit subgroups for all finite unramified extensions $K_d|K$, the limit being with respect to the norm maps. Writing k_d for the residual field of K_d, we have

$$I/P\cong\varprojlim_N U_{K_d}/U^1_{K_d}=\varprojlim_N k_d^\times,$$

which induces a canonical isomorphism of the respective groups of characters:

$$X(I/P)\cong X\Big(\varprojlim_N k_d^\times\Big)=\mathcal{X}.$$

Thus we may (and do) identify $\mathcal{X} := X(I/P)$.

Now let $\mathrm{Irr}(W_K/P)$ denote the set of all irreducible "tame" representations of W_K. The I-equivalence of $\rho, \rho' \in \mathrm{Irr}(W_K/P)$ means that the restrictions $\rho|_I$ and $\rho'|_I$ are equivalent. In particular, for $\chi \in X(I/P)$ consider the I-equivalence class of $\rho(\chi)$, where $\rho(\chi)$ is any irreducible representation of W_K/P such that $\chi \subset \rho(\chi)|_I$. Then we have the bijective mapping

$$\Gamma\backslash\mathcal{X} \longrightarrow \mathrm{Irr}(W_K/P)_I \tag{4.1}$$

such that $\mathcal{X} \ni \chi \longmapsto \rho(\chi)|_I$. We write $\rho(\chi)_I$ for the I-equivalence class of $\rho(\chi)$. From (4.1) we obtain the bijection

$$P(\mathcal{X})^\Gamma \ni \lambda \longmapsto \mu_\lambda \in \mathrm{Rep}\big(W'_K/P\big)_I, \tag{4.2}$$

where

$$\mu_\lambda := \bigoplus_{[\chi]\in\Gamma\backslash\mathcal{X}} \mathrm{sp}_K(\lambda_\chi) \otimes \rho(\chi)_I,$$

which determines an I-equivalence class of representations of W'_K/P. Combining the inverse of (3.1) with (4.2), we obtain Macdonald's bijective mapping

$$\mathcal{M}(k): \Omega \xrightarrow{(3.1)} P(\mathcal{X})^\Gamma \xrightarrow{(4.2)} \mathrm{Rep}\big(W'_K/P\big)_I,$$

where $\tau_\lambda \longmapsto \mu_\lambda$ for all $\lambda \in P(\mathcal{X})^\Gamma$.

Observe that

$$n = \dim(\mu_\lambda) = \sum_{[\chi]\in\Gamma\backslash\mathcal{X}} \big|[\chi]\big||\lambda_\chi| \quad\Longleftrightarrow\quad \Omega_n \ni \tau_\lambda = \prod_{[\chi]\in\Gamma\backslash\mathcal{X}} \pi(\chi)^{\lambda_\chi}.$$

We also see that the determinant character of μ_λ, considered as a function of $k^\times = U_K/U_K^1 \hookrightarrow (W_K/P)^{\mathrm{ab}}$, is the same as the central character of τ_λ [Mac1, (1.2)]. Moreover, the cuspidal support of τ_λ (refer to the last part of Section 2) is the divisor

$$D(\tau_\lambda) = \sum_{[\chi]\in\Gamma\backslash\mathcal{X}} |\lambda_\chi|\pi(\chi).$$

The restriction to I/P of the representation $\mu_\lambda \in \mathrm{Rep}(W'_K/P)_I$ also corresponds naturally to this divisor.

5. An expanded version of diagram (1.1)

We begin by factoring the diagram (1.1) and we assert that the factored or expanded diagram is also commutative:

5.1. Theorem. *Let $K_d|K$ be an unramified extension of local fields lying over the extension of finite fields $k_d|k$. In the diagram* (5.1) *below let "bc" denote Shintani base change,* $\iota: P(\mathcal{X})^{\Gamma_k} \hookrightarrow$

$P(\mathcal{X})^{\Gamma_{k_d}}$ the canonical injection, and "res" the restriction mapping for representations of Weil–Deligne groups. Then the diagram

$$\begin{array}{ccccc} \Omega(k) & \longleftarrow & P(\mathcal{X})^{\Gamma_k} & \longrightarrow & \mathrm{Rep}(W'_K/P)_I \\ \downarrow{\scriptstyle \mathrm{bc}} & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \mathrm{res}} \\ \Omega(k_d) & \longleftarrow & P(\mathcal{X})^{\Gamma_{k_d}} & \longrightarrow & \mathrm{Rep}(W'_{K_d}/P)_I \end{array} \tag{5.1}$$

is commutative.

We deal with the right side of the diagram immediately:

5.2. Lemma. *The right square of diagram* (5.1) *is commutative.*

Proof. On the one hand, it is clear that

$$\mathrm{res}(\mu_\lambda) = \mathrm{res}\left(\bigoplus_{[\chi]\in\Gamma\backslash\mathcal{X}} \mathrm{sp}_K(\lambda_\chi)\otimes\rho(\chi)_I\right) = \bigoplus_{[\chi]\in\Gamma\backslash\mathcal{X}} \mathrm{res}\big(\mathrm{sp}_K(\lambda_\chi)\otimes\rho(\chi)_I\big).$$

On the other, if we consider $\lambda \in P(\mathcal{X})^{\Gamma_k} \subset P(\mathcal{X})^{\Gamma_{k_d}}$ and form the corresponding representation on the k_d-level, then we obtain:

$$\mu_{K_d,\lambda} = \bigoplus_{[\![\psi]\!]\in\Gamma_{k_d}\backslash\mathcal{X}} \mathrm{sp}_{K_d}(\lambda_\psi)\otimes\rho_{K_d}(\psi)_I = \bigoplus_{[\chi]\in\Gamma\backslash\mathcal{X}}\left(\bigoplus_{[\![\psi]\!]\subset[\chi]} \mathrm{sp}_{K_d}(\lambda_\psi)\otimes\rho_{K_d}(\psi)_I\right),$$

where we have written $[\![\psi]\!] := \Gamma_{k_d}\psi$. We must show that

$$\mathrm{res}\big(\mathrm{sp}_K(\lambda_\chi)\otimes\rho(\chi)_I\big) = \bigoplus_{[\![\psi]\!]\subset[\chi]} \mathrm{sp}_{K_d}(\lambda_\psi)\otimes\rho_{K_d}(\psi)_I.$$

However, for $[\![\psi]\!] \subset [\chi]$, we have $\lambda_\psi = \lambda_\chi = \delta$ and $\mathrm{res}(\mathrm{sp}_K(\delta)) = \mathrm{sp}_{K_d}(\delta)$, since $\|w\|_K = \|w\|_{K_d}$ for $w \in W_{K_d} \subset W_K$. Thus we have only to show that

$$\mathrm{res}\big(\rho(\chi)_I\big) = \bigoplus_{[\![\psi]\!]\subset[\chi]} \rho_{K_d}(\psi)_I,$$

which is obvious, since $W_K \supset W_{K_d} \supset I$. □

The rest of the paper will be devoted to showing that the left square of (5.1) is also commutative. Obviously this will imply that (1.1) is commutative. In the next three sections we set the stage for our proof.

6. Shintani descent and base change

We review some definitions and facts from Shintani "descent" or "base change" theory. For details the reader may consult [Sh] and [SZ2, §B].

Let G be a finite group and t an automorphism of G of finite order e. Set $\tilde{G} := G \rtimes \langle t\rangle$, the semidirect product, and let $G^t \subset G$ be the subgroup of G consisting of all t-fixed points.

We call the pair (G,t) *standard* if: (i) there exists a bijection $Gt/{\sim_G} \longleftrightarrow G^t/{\sim_{G^t}}$ between the set of conjugacy classes of $\tilde{G}$ with support in the coset Gt and the set of conjugacy classes of G^t such that $Gt \supset [gt] \longmapsto [(gt)^e] \cap G^t$, the latter being a conjugacy class of G^t; and,

(ii) with respect to normalized measures, the induced bijection

$$\mathrm{Sh}_t : \mathcal{Z}(Gt) \longrightarrow \mathcal{Z}\big(G^t\big) \qquad f \longmapsto \mathrm{Sh}_t(f) \qquad f\big([gt]\big) = \mathrm{Sh}_t(f)\big(\big[(gt)^e\big]\big)$$

from the set of G class functions on the coset Gt to the set of G^t class functions on G^t is an isometry of normalized, complex inner product spaces.

The map inverse to Sh_t is

$$\mathrm{bc}_t : \mathcal{Z}\big(G^t\big) \longrightarrow \mathcal{Z}(Gt) \qquad \varphi \longmapsto \mathrm{bc}_t(\varphi) \qquad \mathrm{bc}_t(\varphi)\big([gt]\big) = \varphi\big(\big[(gt)^e\big]\big).$$

We note that $\mathrm{bc}_t(\varphi)([t]) = \varphi(1)$; thus $\mathrm{bc}_t(\varphi)([t]) > 0$ if φ is a character.

The inner product for the space of functions on the coset Gt is given by the formula

$$\langle f_1, f_2\rangle_{Gt} = \frac{1}{|G|}\sum_{x\in G} f_1(xt)\overline{f_2(xt)} \quad \big(f_1, f_2 \in \mathcal{Z}(Gt)\big).$$

Let $G^{\wedge,t}$ denote the set of irreducible t-invariant characters of G.

6.1. Lemma. *For each $\theta \in G^{\wedge,t}$ fix an extension $\hat{\theta} \in \tilde{G}^{\wedge}$.*

(i) *Set $S := \{\hat{\theta}|_{Gt}\colon \theta \in G^{\wedge,t}\}$. Then S is an orthonormal basis of $\mathcal{Z}(Gt)$.*
(ii) *For any $\theta \in G^{\wedge,t}$ and any true character $\hat{\rho}$ of $\tilde{G}$ such that $\langle\hat{\rho}|_G, \theta\rangle = 1$ the inner product $\langle\hat{\rho}|_{Gt}, \hat{\theta}|_{Gt}\rangle_{Gt}$ is an eth root of unity.*

Proof. (i) See [Sh, 1-1, 1-2] and [SZ2, §B1.1].

(ii) For ζ an eth root of unity let $\hat{\lambda}_\zeta$ denote the inflation to $\tilde{G}$ of the character λ_ζ of the cyclic group $\langle t\rangle$ such that $\lambda_\zeta(t) = \zeta$. By Frobenius, we have

$$1 = \langle\hat{\rho}|_G, \theta\rangle_G = \big\langle\hat{\rho}, \mathrm{ind}_G^{\tilde{G}}(\theta)\big\rangle_{\tilde{G}} = \sum_{\zeta\colon \zeta^e=1} \langle\hat{\rho}, \hat{\lambda}_\zeta\hat{\theta}\rangle_{\tilde{G}},$$

which implies that there is a unique eth root of unity ζ_0 such that $\hat{\rho} = \hat{\lambda}_{\zeta_0}\hat{\theta} + \hat{\rho}_1$ and $\langle\hat{\rho}_1, \hat{\lambda}_\zeta\hat{\theta}\rangle_{\tilde{G}} = 0$ for all ζ such that $\zeta^e = 1$. Therefore, by (i),

$$\langle\hat{\rho}|_{Gt}, \hat{\theta}|_{Gt}\rangle_{Gt} = \big\langle(\hat{\lambda}_{\zeta_0}\hat{\theta})|_{Gt}, \hat{\theta}|_{Gt}\big\rangle_{Gt} = \zeta_0. \qquad \square$$

We return to $k_d|k$, an extension of the finite field k of degree d and we let ϕ be a generator of $\mathrm{Gal}(k_d|k)$. Let $\phi = t$, $d = e$, and let $\tilde{G}_n(k_d) := G_n(k_d) \rtimes \langle\phi\rangle$. Then for all $n \geqslant 1$, $G_n =$

$G_n(k) = G_n(k_d)^\phi$ and the pair $(G_n(k_d), \phi)$ is a standard pair (see above). Similarly, if $P(k_d)$ is any parabolic subgroup of $G_n(k_d)$ which is defined over k, then $(P(k_d), \phi)$ is also a standard pair with $P(k_d)^\phi = P(k)$ (cf. [Sh]).

Let $\mathcal{Z}_{\phi,n} := \mathcal{Z}(G_n(k_d)\phi)$ denote the space of complex-valued class functions on $\tilde{G}_n(k_d)$ with support in $G_n(k_d)\phi$ and $\mathcal{Z}_n := \mathcal{Z}(G_n)$ the space of class functions on G_n. Let $\Omega_n(k_d)^\phi$ denote the set of all ϕ-invariant irreducible characters of $G_n(k_d)$. Every irreducible character $\tau \in \Omega_n(k_d)^\phi$ has a unique extension $\tilde{\tau}$ to $\tilde{G}_n(k_d)$ such that $\tilde{\tau}(\phi) > 0$ (see [SZ2, B3.3(i)]). We call $\tilde{\tau}$ the *canonical extension* of τ. We write $\tilde{\Omega}_n$ for the set of canonical extension characters of $\tilde{G}_n(k_d)$. The mapping $\tau \longmapsto \tilde{\tau}$ is a bijection of $\Omega_n(k_d)^\phi$ to $\tilde{\Omega}_n$, and from 6.1(i) we see that the restriction mapping

$$\tilde{\tau} \longmapsto \tilde{\tau}_\phi := \tilde{\tau}|_{G_n(k_d)\phi} \tag{6.1}$$

is injective and maps $\tilde{\Omega}_n$ to an orthonormal basis, to be denoted $\tilde{\Omega}_{\phi,n}$, of $\mathcal{Z}_{\phi,n}$.

Since $(G_n(k_d), \phi)$ is a standard pair, the mapping

$$\mathrm{Sh}_\phi : \mathcal{Z}_{\phi,n} \longrightarrow \mathcal{Z}_n, \qquad \mathrm{Sh}_\phi(f)\big(\big[(g\phi)^d\big]\big) = f(g\phi) \quad \big(g \in G_n(k_d)\big)$$

is an isometry of complex inner product spaces. Shintani's fundamental theorem [Sh, Theorem 1], with some additions remarked in [SZ2, §B], asserts that the composition of the mapping Sh_ϕ with the natural bijection $\Omega_n(k_d)^\phi \longrightarrow \tilde{\Omega}_{\phi,n}$ defines a bijection

$$\mathrm{Sh}_{k_d\downarrow k} : \Omega_n(k_d)^\phi \longrightarrow \Omega_n, \qquad \mathrm{Sh}_{k_d\downarrow k}(\tau) := \mathrm{Sh}_\phi(\tilde{\tau}_\phi).$$

We also have the inverse map $\mathrm{bc}_{k\uparrow k_d} : \Omega_n \ni \pi \longmapsto \tau \in \Omega_n(k_d)^\phi$, which is defined by the condition that $\tilde{\tau}_\phi = \mathrm{bc}_\phi(\pi)$. We call $\mathrm{bc}_{k\uparrow k_d}$ the *Shintani base change* mapping. The mappings $\mathrm{Sh}_{k_d\downarrow k}$ and $\mathrm{bc}_{k\uparrow k_d}$ are independent of the choice of the generator ϕ of $\mathrm{Gal}(k_d|k)$ (cf. [SZ2, B3.2]).

7. The coset algebra

In this section we want to construct the coset PSH algebra R_ϕ and to prove that Shintani base change defines a natural isomorphism of R_ϕ to R.

Formally we set $G_0(k_d) := 1$, $\tilde{G}_0(k_d) := \langle\phi\rangle$, and therefore $\mathcal{Z}_{\phi,0} := \mathbb{C}$, the one-dimensional space with basis the 1-valued function on $G_0(k_d)\phi$. This basis element comprises the singleton set $\tilde{\Omega}_{\phi,0}$.

Set

$$\tilde{\Omega}_\phi := \coprod_{n\geqslant 0} \tilde{\Omega}_{\phi,n}, \qquad \mathcal{Z}_\phi := \bigoplus_{n\geqslant 0} \mathcal{Z}_{\phi,n} \quad \text{and} \quad \mathcal{Z} := \bigoplus_{n\geqslant 0} \mathcal{Z}_n.$$

Then $\mathcal{Z}$ is a graded $\mathbb{C}$-vector space and the quadruple $(\mathcal{Z}, m, m^*, \Omega)$ is a PSH algebra, the extension of scalars of the Hopf algebra already discussed in Section 2 (see [SpZ, 1.2]).

We want to define on $\mathcal{Z}_\phi$ a PSH algebra structure which is naturally isomorphic to that of $\mathcal{Z}$. We know that $\mathcal{Z}_\phi$ is a graded $\mathbb{C}$-vector space with an inner product space structure. Moreover, $\tilde{\Omega}_\phi$ is an orthonormal basis for $\mathcal{Z}_\phi$ such that for all n $\tilde{\Omega}_{\phi,n}$ is a basis for $\mathcal{Z}_{\phi,n}$ and Sh_ϕ maps $\tilde{\Omega}_{\phi,n}$ isometrically to Ω_n. Therefore, as graded inner product spaces, the spans of these two spaces are canonically isomorphic.

We use a variant of the definitions given in [SpZ, 1.2] to introduce a multiplication and comultiplication:

$$m_\phi : \mathcal{Z}_\phi \otimes \mathcal{Z}_\phi \longrightarrow \mathcal{Z}_\phi \qquad m_\phi^* : \mathcal{Z}_\phi \longrightarrow \mathcal{Z}_\phi \otimes \mathcal{Z}_\phi.$$

It suffices to give these definitions for homogeneous elements; we shall define

$$m_{\phi;r,s} : \mathcal{Z}_{\phi,r} \otimes \mathcal{Z}_{\phi,s} \longrightarrow \mathcal{Z}_{\phi,r+s}$$

and

$$m_\phi^{*,n} : \mathcal{Z}_{\phi,n} \longrightarrow \bigoplus_{r,s:r+s=n} \mathcal{Z}_{\phi,r} \otimes \mathcal{Z}_{\phi,s}.$$

Let us consider first the definition of $m_{\phi;r,s}$. Let $P = L \ltimes U_P \subset G_{r+s}$ be a k parabolic subgroup with Levi subgroup $L \cong G_r \times G_s$ and unipotent radical U_P. Let $f_r \otimes f_s \in \mathcal{Z}_{\phi,r} \otimes \mathcal{Z}_{\phi,s}$ and consider $f_r \otimes f_s$ as inflated to $\tilde{P}(k_d) := P(k_d) \rtimes \langle\phi\rangle$: $f_r \otimes f_s((x,y)u\phi) = f_r(x\phi) \otimes f_s(y\phi)$ for $(x,y) \in L(k_d)$ and $u \in U_P(k_d)$. Extend $f_r \otimes f_s$ by zero to $\tilde{G}_{r+s}(k_d)$ and define, using Frobenius's formula, the (parabolically) induced class function on $G_{r+s}(k_d)\phi$:

$$m_\phi(f_r \otimes f_s)(g\phi) := \frac{1}{|P(k_d)|} \sum_{x \in G_{r+s}(k_d)} (f_r \otimes f_s)\big(x^{-1} g\phi x\big).$$

The support of $f_r \otimes f_s$ lies in the coset $P(k_d)\phi$ and the support of $m_\phi(f_r \otimes f_s)$ in $G_{r+s}(k_d)\phi$.

7.1. Lemma. *Let m denote the multiplication in the Hopf algebra $\mathcal{Z}$ and m_ϕ the multiplication which we have defined for the space $\mathcal{Z}_\phi$. Then:*

$$m_{r,s} \circ (\mathrm{Sh}_\phi \otimes \mathrm{Sh}_\phi) = \mathrm{Sh}_\phi \circ m_{\phi;r,s},$$

where, on the left, the Shintani descent mapping is

$$\mathrm{Sh}_\phi \otimes \mathrm{Sh}_\phi : \mathcal{Z}_{\phi,r} \otimes \mathcal{Z}_{\phi,s} \longrightarrow \mathcal{Z}_r \otimes \mathcal{Z}_s \tag{7.1}$$

and, on the right, the Shintani descent mapping is

$$\mathrm{Sh}_\phi : \mathcal{Z}_{\phi,r+s} \longrightarrow \mathcal{Z}_{r+s}.$$

Proof. First we note that $\mathcal{Z}_{\phi,r} \otimes \mathcal{Z}_{\phi,s} \cong \mathcal{Z}(L(k_d)\phi)$, so we may regard the mapping $\mathrm{Sh}_\phi \otimes \mathrm{Sh}_\phi$ of (7.1) as $\mathrm{Sh}_\phi : \mathcal{Z}(L(k_d)\phi) \longrightarrow \mathcal{Z}(L(k))$. It suffices to show that the following diagram is commutative, where the vertical arrows labeled "inf" denote inflation mappings from $L(k_d)\phi$ to $P(k_d)\phi$, respectively, $L(k)$ to $P(k)$ and the arrows labeled "ind" denote the respective parabolic induction mappings.

$$\begin{array}{ccc} \mathcal{Z}(L(k_d)\phi) & \xrightarrow{\mathrm{Sh}_\phi} & \mathcal{Z}(L(k)) \\ \downarrow \mathrm{inf} & & \downarrow \mathrm{inf} \\ \mathcal{Z}(P(k_d)\phi) & \xrightarrow{\mathrm{Sh}_\phi} & \mathcal{Z}(P(k)) \\ \downarrow \mathrm{ind} & & \downarrow \mathrm{ind} \\ \mathcal{Z}_{\phi,r+s} & \xrightarrow{\mathrm{Sh}_\phi} & \mathcal{Z}_{r+s} \end{array}$$

Since $(P(k_d),\phi)$ is a standard pair, [SZ2, B1.4(iii)] implies that the lower square of the diagram is commutative; in other words Shintani descent commutes with parabolic induction.

Let us now prove that the upper square of the diagram is commutative. Let $\mathcal{Z}(P(k_d)\phi, U_P)$ denote the subspace of $\mathcal{Z}(P(k_d)\phi)$ consisting of all class functions which satisfy $f(p\phi u) = f(up\phi) = f(p\phi)$ for all $p \in P(k_d), u \in U_P(k_d)$ and, similarly, write $\mathcal{Z}(P(k), U_P)$ for the subspace of $\mathcal{Z}(P(k))$ which consists of all functions which are left and right $U_P(k)$-invariant. Obviously the restriction mappings

$$\mathrm{res}: \mathcal{Z}\big(P(k_d)\phi, U_P\big) \longrightarrow \mathcal{Z}\big(L(k_d)\phi\big) \quad \text{and} \quad \mathrm{res}: \mathcal{Z}\big(P(k), U_P\big) \longrightarrow \mathcal{Z}\big(L(k)\big) \tag{7.2}$$

are injective mappings. Note that

$$u^{-1}x\phi u = u^{-1}\big(x\phi u\phi^{-1}x^{-1}\big)x\phi \quad \big(u \in U_P(k_d), x \in L(k_d)\big).$$

Since ϕ and $L(k_d)$ both normalize $U_P(k_d)$ and $U_P(k_d) \cap L(k_d) = (1)$, we see that if $u^{-1}x\phi u \in L(k_d)\phi$ $(x \in L(k_d))$, then $u^{-1}x\phi u = x\phi$; thus, two elements in $L(k_d)\phi$ are in the same $P(k_d)$ class if and only if they are in the same $L(k_d)$ class. Similarly two elements in $L(k)$ are conjugate in $L(k)$ if and only if they are conjugate in $P(k)$. Therefore, the restriction mappings of (7.2) are bijections and the respective mappings

$$\mathrm{inf}: \mathcal{Z}\big(L(k_d)\phi\big) \longrightarrow \mathcal{Z}\big(P(k_d)\phi, U_P\big) \quad \text{and} \quad \mathrm{inf}: \mathcal{Z}\big(L(k)\big) \longrightarrow \mathcal{Z}\big(P(k), U_P\big)$$

are canonical bijections which are inverses of the respective restriction mappings. Since $(P(k_d),\phi)$ and $(L(k_d),\phi)$ are standard pairs, Sh_ϕ commutes with restriction (see [SZ2, B1.4(i)]). It follows that for every $f \in \mathcal{Z}(P(k_d)\phi, U_P)$

$$\mathrm{res}^{L(k)}_{P(k)} \circ \mathrm{Sh}_\phi(f) = \mathrm{Sh}_\phi \circ \mathrm{res}^{L(k_d)\phi}_{P(k_d)\phi}(f). \tag{7.3}$$

If we take $f = \mathrm{inf}^{P(k_d)\phi}_{L(k_d)\phi}(g)$ and apply $\mathrm{inf}^{P(k)}_{L(k)}$ to both sides of (7.3), then we obtain

$$\mathrm{Sh}_\phi \circ \mathrm{inf}^{P(k_d)\phi}_{L(k_d)\phi}(g) = \mathrm{inf}^{P(k)}_{L(k)} \circ \mathrm{Sh}_\phi(g),$$

so the upper square of the diagram commutes too. □

Next we consider the comultiplication $m_\phi^{*,n} := \bigoplus_{r,s:r+s=n} m_{\phi;r,s}^*$ on the coset $G_n(k_d)\phi$. For $f \in \mathcal{Z}_{\phi,n}$ we define

$$m_{\phi;r,s}^{*,n}(f)(x\phi) := \frac{1}{|U_P(k_d)|} \sum_{u \in U_P(k_d)} f(ux\phi) \quad \bigl(x \in L(k_d),\ u \in U_P(k_d)\bigr).$$

The mapping $m_{\phi;r,s}^{*,n}$ is the Harish-Chandra/Jacquet functor from class functions on the coset $G_n(k_d)\phi$ to class functions on $L(k_d)\phi$.

7.2. Lemma. *Let m^* denote the comultiplication in the Hopf algebra $\mathcal{Z}$ and m_ϕ^* the comultiplication which we have defined for the space $\mathcal{Z}_\phi$. Then*:

$$(\mathrm{Sh}_\phi \otimes \mathrm{Sh}_\phi) \circ m_{\phi;r,s}^{*,n} = m_{r,s}^{*,n} \circ \mathrm{Sh}_\phi .$$

Proof. For the proof we have to verify that the following diagram is commutative:

$$\begin{array}{ccc}
\mathcal{Z}_{\phi,r+s} & \xrightarrow{\mathrm{Sh}_\phi} & \mathcal{Z}_{r+s} \\
\Big\downarrow \mathrm{res}^{P(k_d)\phi}_{G_{r+s}(k_d)\phi} & & \Big\downarrow \mathrm{res}^{P(k)}_{G_{r+s}(k)} \\
\mathcal{Z}(P(k_d)\phi) & \xrightarrow{\mathrm{Sh}_\phi} & \mathcal{Z}(P(k)) \\
\Big\downarrow \Pi_{U_P} & & \Big\downarrow \Pi_{U_P} \\
\mathcal{Z}(P(k_d)\phi, U_P) & \xrightarrow{\mathrm{Sh}_\phi} & \mathcal{Z}(P(k), U_P) \\
\Big\downarrow \mathrm{res}^{L(k_d)\phi}_{P(k_d)\phi} & & \Big\downarrow \mathrm{res}^{L(k)}_{P(k)} \\
\mathcal{Z}(L(k_d)\phi) & \xrightarrow{\mathrm{Sh}_\phi} & \mathcal{Z}(L(k)).
\end{array}$$

The vertical arrows Π_{U_P} denote the orthogonal projection operators on the image spaces. The top square of the diagram is commutative, because Sh_ϕ commutes with restriction. We have already seen that the bottom square of the diagram is commutative; in fact, the vertical restriction maps are bijective. Only the commutativity of the middle diagram needs to be verified. However, both middle Sh_ϕ-maps are isometric isomorphisms, which implies that Sh_ϕ maps the orthogonal complement of $\mathcal{Z}(P(k_d)\phi, U_P)$ isometrically to the orthogonal complement of $\mathcal{Z}(P(k), U_P)$. It follows that the middle square is commutative too. □

From the preceding two lemmas we obtain:

7.3. Proposition. *The graded degree-preserving map $\mathrm{Sh}_\phi : \mathcal{Z}_\phi \xrightarrow{\sim} \mathcal{Z}$ defines an isomorphism of PSH algebras which takes the orthonormal basis $\tilde{\Omega}_\phi = \coprod_{n \geqslant 0} \tilde{\Omega}_{\phi,n}$ to the orthonormal basis $\Omega = \coprod_{n \geqslant 0} \Omega_n$. In particular Sh_ϕ induces an isomorphism between the $\mathbb{Z}$-subalgebras R_ϕ and R which are spanned over $\mathbb{Z}$ by $\tilde{\Omega}_\phi$ and Ω, respectively.*

Proof. We know that Sh_ϕ is an isometry of inner product spaces which takes the orthonormal basis $\tilde{\Omega}_{\phi,n}$ bijectively to the orthonormal basis Ω_n for all $n \geqslant 0$. This implies that Sh_ϕ is also a bijection of $\tilde{\Omega}_\phi$ to Ω which respects the graded structures of $\mathcal{Z}_\phi$ and $\mathcal{Z}$. Identifying $\mathcal{Z}_\phi$ and $\mathcal{Z}$ via the bijection $\mathrm{Sh}_\phi : \tilde{\Omega}_\phi \xrightarrow{\sim} \Omega$, we know that Sh_ϕ commutes with the respective multiplications and comultiplications (7.1 and 7.2). From [SpZ] we know that $\mathcal{Z}$ is a PSH algebra. Since the respective multiplications and comultiplications commute with Sh_ϕ^{-1}, we may transport the PSH algebra structure $(\mathcal{Z}, m, m^*, \Omega)$ to $(\mathcal{Z}_\phi, m_\phi, m_\phi^*, \tilde{\Omega}_\phi)$. Finally, the $\mathbb{Z}$-module morphism $\mathrm{Sh}_\phi : R_\phi \xrightarrow{\sim} R$ is a Hopf algebra isomorphism, since the orthonormal bases $\tilde{\Omega}_\phi$ and Ω correspond under this mapping and the multiplications and comultiplications commute with Sh_ϕ and stabilize R_ϕ and R and their tensor powers. □

The following is included for use in the proof of 9.3:

7.4. Lemma. *Let $\mu_i \in \Omega_{m_i}(k_d)^\phi$ for $i = 1, \ldots, p$ and assume that the Hopf product $U = \mu_1 \cdots \mu_p \in \Omega_n(k_d)$. Then $U \in \Omega_n(k_d)^\phi$ and $\tilde{U}_\phi = \tilde{\mu_1}_\phi \cdots \tilde{\mu_p}_\phi \in \tilde{\Omega}_\phi$.*

Proof. Let $\tilde{P}(k_d) := P(k_d) \rtimes \langle\phi\rangle$, where $P(k_d) = (G_{m_1}(k_d) \times \cdots \times G_{m_p}(k_d)) \ltimes U_P(k_d)$. Let $\tilde{\mu}_1, \ldots, \tilde{\mu}_p$ be the canonical extensions of $\mu_1, \ldots, \mu_p$, so that the restrictions to the ϕ-coset $\tilde{\mu_1}_\phi, \ldots, \tilde{\mu_p}_\phi$ each satisfy $\tilde{\mu}_i(\phi) = \tilde{\mu_i}_\phi(\phi) > 0$ $(1 \leqslant i \leqslant p)$. Then, computing the Hopf product in R_ϕ, we obviously have $(\tilde{\mu_1}_\phi \cdots \tilde{\mu_p}_\phi)(\phi) > 0$. Since U is irreducible, $\tilde{U} := \mathrm{ind}_{\tilde{P}(k_d)}^{\tilde{G}(k_d)} \tilde{\mu}_1 \otimes \cdots \otimes \tilde{\mu}_p$ is a ϕ-invariant irreducible character of $\tilde{G}$ which extends U. Thus $\tilde{U}_\phi = \tilde{\mu_1}_\phi \cdots \tilde{\mu_p}_\phi \in \tilde{\Omega}_\phi$. □

8. The Hopf norm

For $\sigma \in \mathcal{C}(k_d)$ we want to define the *Hopf norm* of an element of $R(\sigma)$ (see Section 2). Unless we say otherwise all "products" in this section will be Hopf products.

Fix $\sigma \in \mathcal{C}(k_d)$ and let $\tau = \sigma^\lambda \in \Omega(\sigma)$ for some nonzero partition λ (see 2.1). We set

$$\mathcal{N}(\tau) = \mathcal{N}_{k_d|k}(\tau) := \prod_{\tau' \in \mathrm{Gal}(k_d|k)\tau} \tau'.$$

8.1 implies that the number of factors τ' in the Hopf product $\mathcal{N}(\tau)$ is $l(\sigma) := |\mathrm{Gal}(k_d|k)\sigma|$; obviously, $l(\sigma) \mid d$.

8.1. Lemma. *Let $\sigma \in \mathcal{C}(k_d)$ and $\psi \in \mathrm{Gal}(k_d|k)$. Then ${}^\psi\sigma \in \mathcal{C}(k_d)$, ${}^\psi\Omega(\sigma) = \Omega({}^\psi\sigma)$ (see 2.1(i)), and ψ induces an isomorphism $\psi : R(\sigma) \longrightarrow R({}^\psi\sigma)$ of primary PSH algebras such that $x_i({}^\psi\sigma) = {}^\psi x_i(\sigma)$ for all $i \geqslant 1$. Moreover, ${}^\psi(\sigma^\lambda) = ({}^\psi\sigma)^\lambda$ for every nonzero partition λ and*

$$\mathcal{N}(\sigma^\lambda) = \prod_{\sigma' \in \mathrm{Gal}(k_d|k)\sigma} \sigma'^\lambda. \tag{8.1}$$

For all $\tau \in \Omega(\sigma)$ the length of the Galois orbit satisfies $|\mathrm{Gal}(k_d|k)\tau| = |\mathrm{Gal}(k_d|k)\sigma|$ and $\mathcal{N}(\tau)$ is irreducible and $\mathrm{Gal}(k_d|k)$-invariant.

Proof. The action of ψ on $R(k_d)$ is invertible and it commutes with m and m^*, so ψ induces bijections ${}^\psi\Omega(\sigma) = \Omega({}^\psi\sigma)$ and $\psi : R(\sigma) \longrightarrow R({}^\psi\sigma)$. These bijections define an isomorphism of primary PSH algebras. Moreover, if $\rho \in \Omega(\sigma)$ admits a Whittaker vector, then the same is true of ${}^{\psi}\rho \in \Omega({}^{\psi}\sigma)$, so, for all $i \geqslant 1$, $\psi(x_i(\sigma)) = x_i({}^{\psi}\sigma)$; thus, $\psi : R(\sigma) \longrightarrow R({}^{\psi}\sigma)$ is the isomorphism which sends GS characters to GS characters. For any nonzero partition λ the character σ^{λ} is a polynomial $p(\lambda, \{x_i(\sigma)\}) \in \mathbb{Z}[\{x_i(\sigma)\}]$ and $\psi(p(\lambda, \{x_i(\sigma)\})) = p(\lambda, \{x_i({}^{\psi}\sigma)\}) \in \mathbb{Z}[\{x_i({}^{\psi}\sigma)\}]$. Therefore, ${}^{\psi}(\sigma^{\lambda}) = ({}^{\psi}\sigma)^{\lambda}$. The character $\mathcal{N}(\sigma^{\lambda})$ of (8.1) is irreducible because it is a product of irreducible characters with disjoint cuspidal supports and it is $\mathrm{Gal}(k_d|k)$-invariant because the Galois action permutes the factors and the product is commutative; obviously, $|\mathrm{Gal}(k_d|k)\sigma^{\lambda}| = |\mathrm{Gal}(k_d|k)\sigma|$ too. Since every $\tau \in \Omega(\sigma)$ is of the form σ^{λ} for some nonzero partition, the lemma follows. $\square$

Noting that the $\mathrm{Gal}(k_d|k)$-action on $\Omega(k_d)$ induces a PSH algebra automorphism of $R(k_d)$ which permutes PSH primary subalgebras, we see that the norm mapping defined above extends to all elements $\rho = \sum_{\lambda \neq (0)} m_{\lambda}\sigma^{\lambda} \in R(\sigma)$. Thus

$$\mathcal{N}(\rho) := \prod_{\rho' \in \mathrm{Gal}(k_d|k)\rho} \rho' = \prod_{\sigma' \in \mathrm{Gal}(k_d|k)\sigma} \left(\sum_{\lambda \neq (0)} m_{\lambda} \sigma'^{\lambda} \right) \quad \in R(k_d)^{\phi}$$

is well defined and, moreover, $\mathcal{N}(\rho_1\rho_2) = \mathcal{N}(\rho_1)\mathcal{N}(\rho_2)$ for any $\rho_1, \rho_2 \in R(\sigma)$.

8.2. Lemma. *Let $\sigma \in \mathcal{C}(k_d)$, let $\lambda = (\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_r > 0)$ be a nonzero partition of n, and consider $x_{\lambda}(\sigma) = \prod_{i=1}^{r} x_{\ell_i}(\sigma)$. Write $x_{\lambda}(\sigma) = \sum_{\mu : |\mu| = n} m_{\mu}(\lambda)\sigma^{\mu}$, where $\Omega_{nd(\sigma)}(k_d) = \{\sigma^{\mu}\}_{|\mu|=n}$ and the multiplicities $m_{\mu}(\lambda) \geqslant 0$ depend upon λ. Then the* $\mathrm{Gal}(k_d|k)$*-invariant irreducible components of $\mathcal{N}(x_{\lambda}(\sigma))$ are the components of the form $\mathcal{N}(\sigma^{\mu})$ and the multiplicity of $\mathcal{N}(\sigma^{\mu})$ in $\mathcal{N}(x_{\lambda}(\sigma))$ is $m_{\mu}(\lambda)^{l(\sigma)}$ ($l(\sigma) = |\mathrm{Gal}(k_d|k)\sigma|$).*

Proof. The assertion is immediate by applying the distributive law to Hopf multiplication. Obviously only the Hopf norms of characters σ^{λ} can be $\mathrm{Gal}(k_d|k)$-invariant. $\square$

9. The reduction of the proof of Theorem 5.1 to a key lemma

From Green's construction of the cuspidal characters of finite general linear groups we have for every $d \geqslant 1$ the parametrization (cf. 3.1(ii); note that the group $\mathcal{X}$ does not depend upon $d \geqslant 1$):

$$\Gamma_{k_d} \backslash \mathcal{X} \longleftrightarrow \mathcal{C}(k_d), \qquad \chi \longmapsto \pi_{k_d}(\chi). \tag{9.1}$$

This correspondence is compatible with the action of Γ on $\mathcal{C}(k_d)$:

$$\pi_{k_d}(\chi^{\gamma}) = \pi_{k_d}(\chi)^{\bar{\gamma}} \quad \big(\gamma \in \Gamma,\ \bar{\gamma} = \mathrm{res}_{\bar{k}|k_d}(\gamma) \in \mathrm{Gal}(k_d|k)\big), \tag{9.2}$$

so we have the natural bijection

$$\Gamma \backslash \mathcal{X} \longleftrightarrow \mathrm{Gal}(k_d|k) \backslash \mathcal{C}(k_d), \qquad \chi \longmapsto \big[\pi_{k_d}(\chi)\big] = \mathrm{Gal}(k_d|k)\pi_{k_d}(\chi). \tag{9.3}$$

We begin our proof that the left square of (5.1) is commutative by dealing in 9.1 with the cuspidal case; the first assertion of 9.1 is 5.1 for the special case in which $\pi \in \Omega$ is cuspidal.

9.1. Theorem. *Assume that* $\lambda \in P(\mathcal{X})^{\Gamma}$ *is supported on a single* Γ*-orbit* $\Gamma\chi \subset \mathcal{X}$ *and that* $\lambda_\chi = (1)$. *Then* $\tau_\lambda = \pi(\chi) \in \mathcal{C}$ *(see* (3.1) *and* 3.1*) is the cuspidal character with the Green's parameter* $[\chi] = \Gamma\chi$ *and*

$$\mathrm{bc}_{k\uparrow k_d}\big(\pi(\chi)\big) = \prod_{[\![\psi]\!] \in \Gamma_{k_d}\backslash[\chi]} \pi_{k_d}(\psi) = \mathcal{N}\big(\pi_{k_d}(\chi)\big), \tag{9.4}$$

the Hopf product of distinct, irreducible cuspidal characters in $\mathcal{C}(k_d)$. *In other words, the bijective correspondence* $\mathrm{bc}_{k\uparrow k_d}$ *connects the left and right of the correspondence*

$$\mathcal{C} \longleftrightarrow \Gamma\backslash\mathcal{X} \longleftrightarrow \mathrm{Gal}(k_d|k)\backslash\mathcal{C}(k_d),$$

$$\mathcal{C} \ni \pi \longleftrightarrow [\sigma] \in \mathrm{Gal}(k_d|k)\backslash\mathcal{C}(k_d) \quad \Longleftrightarrow \quad \mathrm{bc}_{k\uparrow k_d}(\pi) = \mathcal{N}(\sigma). \tag{9.5}$$

Proof. We refer the reader to [SZ2, 5.7] for an equivalent statement, which is proved using AMT. More general statements may be read out of the cited works of Gyoja and Digne. □

9.2. Corollary. *The set of primitive irreducible elements for the coset algebra* R_ϕ *is the set*

$$\mathcal{C}_\phi := \big\{\tilde{\mathcal{N}}(\sigma)_\phi\colon [\sigma] \in \mathrm{Gal}(k_d|k)\backslash\mathcal{C}(k_d)\big\}, \tag{9.6}$$

where the Hopf norm $\mathcal{N}(\sigma) \in \Omega(k_d)^{\phi}$ *and, as in* (6.1), $\tilde{\mathcal{N}}(\sigma)_\phi$ *denotes the canonical extension of* $\mathcal{N}(\sigma)$ *restricted to* $G\phi$.

Proof. Since $\mathrm{bc}_\phi = \mathrm{Sh}_\phi^{-1}$, 7.3 implies that the mapping $\mathrm{bc}_\phi : R \longrightarrow R_\phi$ is an isomorphism of PSH algebras. Therefore, bc_ϕ maps the set of primitive irreducible elements $\mathcal{C}$ of R bijectively to the set of primitive irreducible elements of R_ϕ. Thus 9.1 implies that $\mathcal{C}_\phi$ is the set of primitive irreducible elements of R_ϕ. □

9.3. Corollary. *Let* $\pi_1, \ldots, \pi_p$ *be distinct elements of* $\mathcal{C}$ *and let* $\tau_i \in \Omega_{m_i}(\pi_i) \subset \Omega$ *for* $i = 1, \ldots, p$ $(m_1 + \cdots + m_p = n)$. *Then* $\tau := \tau_1 \cdots \tau_p \in \Omega_n$ *and*

$$\mathrm{bc}_{k\uparrow k_d}(\tau_1 \cdots \tau_p) = \mathrm{bc}_{k\uparrow k_d}(\tau_1) \cdots \mathrm{bc}_{k\uparrow k_d}(\tau_p). \tag{9.7}$$

Proof. Since the π_i are distinct, the τ_i have disjoint cuspidal support. This implies that τ is irreducible and, by 7.3, $\mathrm{bc}_\phi(\tau) = \mathrm{bc}_\phi(\tau_1) \cdots \mathrm{bc}_\phi(\tau_p) \in \tilde{\Omega}_\phi$. Moreover, $\mathrm{bc}_\phi(\tau_i) = \tilde{\mu}_{i\,\phi}$, where $\mu_i \in \Omega_{m_i}(k_d)^{\phi}$, and, by 9.1, distinct μ_i have disjoint cuspidal support. By definition, $\mu_i = \mathrm{bc}_{k\uparrow k_d}(\tau_i)$ and, because of the disjoint cuspidal support, $U := \mu_1 \cdots \mu_p = \mathrm{bc}_{k\uparrow k_d}(\tau_1) \cdots \mathrm{bc}_{k\uparrow k_d}(\tau_p)$, considered as a Hopf product on $G_n(k_d)$, is irreducible. Now apply 7.4 to deduce that $U = \mathrm{bc}_{k\uparrow k_d}(\tau)$. □

Let $(R_\Delta, m_\Delta, m_\Delta^*, \Omega_\Delta)$ denote the PSH subalgebra of $(R(k_d), m_d, m_d^*, \Omega(k_d))$ generated by the primary PSH subalgebras $R(\sigma)$ for all $\sigma \in \Delta$, where Δ is any set of representatives for $\mathrm{Gal}(k_d|k)\backslash\mathcal{C}(k_d)$. We assume that π and $[\sigma]$ correspond under (9.5). Zelevinsky's structure theory of PSH algebras implies that there are unique isomorphisms of primary PSH algebras

$R(\pi) \longrightarrow R(\sigma)$ such that $x_i(\pi) \longmapsto x_i(\sigma)$ ($\forall i \geqslant 0$) and such that $\pi^\lambda \longmapsto \sigma^\lambda$ for every nonzero partition λ. These isomorphisms induce an isomorphism of PSH algebras

$$\Psi_\Delta : (R, m, m^*, \Omega) \xrightarrow{\sim} \big(R_\Delta, m_\Delta, m^*_\Delta, \Omega_\Delta\big).$$

Composing Ψ_Δ^{-1} with the isomorphism

$$\mathrm{bc}_\phi : (R, m, m^*, \Omega) \xrightarrow{\sim} \big(R_\phi, m_\phi, m^*_\phi, \tilde{\Omega}_\phi\big)$$

of 7.3, we obtain the diagram

$$\begin{array}{ccc} R & \xrightarrow{\Psi_\Delta} & R_\Delta \\ {\scriptstyle \text{identity}}\downarrow & & \downarrow{\scriptstyle \Phi_\Delta} \\ R & \xrightarrow{\mathrm{bc}_\phi} & R_\phi \end{array}$$

where

$$\Phi_\Delta : \big(R_\Delta, m_\Delta, m^*_\Delta, \Omega_\Delta\big) \xrightarrow{\sim} \big(R_\phi, m_\phi, m^*_\phi, \tilde{\Omega}_\phi\big)$$

is the isomorphism of PSH algebras which makes the diagram commutative.

Note that 9.1 implies that $\Phi_\Delta(\sigma) = \mathrm{bc}_\phi(\pi) = \tilde{\mathcal{N}}(\sigma)_\phi$ for all $\sigma \in \Delta$.

9.4. Key Lemma. *For any $\sigma \in \Delta$ and $\pi \in \mathcal{C}$ such that $\mathcal{N}(\sigma) = \mathrm{bc}_{k\uparrow k_d}(\pi)$, the following assertions are equivalent and true*:

(i) $\Phi_\Delta(\rho) = \tilde{\mathcal{N}}(\rho)_\phi$ *for all* $\rho \in \Omega(\sigma)$.
(ii) $\mathrm{bc}_\phi(\pi^\lambda) = \tilde{\mathcal{N}}(\sigma^\lambda)_\phi$ *for all nonzero partitions* λ.
(iii) $\mathrm{bc}_{k\uparrow k_d}(\pi^\lambda) = \mathcal{N}(\sigma^\lambda)$ *for all nonzero partitions* λ.

Proof. We shall prove (ii) in Section 10. Here we check only the equivalence of (i), (ii), and (iii). Since any $\rho \in \Omega(\sigma)$ is of the form σ^λ for some nonzero partition λ and since we may assume that $\Psi_\Delta(\pi) = \sigma$, it follows that the left sides of (i) and (ii) coincide:

$$\mathrm{bc}_\phi\big(\pi^\lambda\big) = \Phi_\Delta\big(\Psi_\Delta\big(\pi^\lambda\big)\big) = \Phi_\Delta\big(\sigma^\lambda\big).$$

The equivalence of (ii) and (iii) follows from the fact that, for every nonzero partition λ, the canonical extension of $\mathrm{bc}_{k\uparrow k_d}(\pi^\lambda)$ on the coset $G(k_d)\phi$ equals $\mathrm{bc}_\phi(\pi^\lambda)$ and, similarly, $\tilde{\mathcal{N}}(\sigma^\lambda)_\phi$ is the canonical extension of $\mathcal{N}(\sigma^\lambda)$ on the coset. □

9.5. Theorem. *For $i = 1, \ldots, r$ let $\pi_i \in \mathcal{C}$, let $\Psi_\Delta(\pi_i) = \sigma_i \in \Delta$, and let λ_i be a nonzero partition. Let $\tau = \pi_1^{\lambda_1} \cdots \pi_r^{\lambda_r}$ be the primary factorization of $\tau \in \Omega$. Then*

$$\mathrm{bc}_{k\uparrow k_d}(\tau) = \mathrm{bc}_{k\uparrow k_d}\big(\pi_1^{\lambda_1}\big) \cdots \mathrm{bc}_{k\uparrow k_d}\big(\pi_r^{\lambda_r}\big) = \mathcal{N}\big(\sigma_1^{\lambda_1}\big) \cdots \mathcal{N}\big(\sigma_r^{\lambda_r}\big).$$

Proof. From (9.7) we obtain the first equal sign and from 9.4(iii) the second. □

9.6. Corollary. *The first square of* (5.1) *is commutative*: *Let* $\lambda \in P(\mathcal{X})^\Gamma$ *and* $\tau_\lambda = \prod_{[\chi]\in\Gamma_k\backslash\mathcal{X}} \pi(\chi)^{\lambda_\chi} \in \Omega$ *as in* (3.1). *Then*

$$\mathrm{bc}_{k\uparrow k_d}(\tau_\lambda) = \prod_{[\chi]\in\Gamma\backslash\mathcal{X}} \mathcal{N}\bigl(\pi_{k_d}(\chi)^{\lambda_\chi}\bigr) = \prod_{[\chi]\in\Gamma\backslash\mathcal{X}} \prod_{[\![\psi]\!]\in\Gamma_{k_d}\backslash[\chi]} \pi_{k_d}(\psi)^{\lambda_\chi}.$$

Proof. By 9.5 it is enough to consider $\lambda \in P(\mathcal{X})^\Gamma$ with support a single orbit $[\chi] \in \Gamma\backslash\mathcal{X}$. In this case, using Eq. (9.4) and Theorem 9.5 we have, for $\tau = \pi(\chi)^{\lambda_\chi}$,

$$\mathrm{bc}_{k\uparrow k_d}(\tau) = \mathcal{N}\bigl(\pi_{k_d}(\chi)^{\lambda_\chi}\bigr) = \prod_{[\![\psi]\!]\in\Gamma_{k_d}\backslash[\chi]} \pi_{k_d}(\psi)^{\lambda_\chi}. \qquad \square$$

10. Proof of the Key Lemma

We assume $\sigma \in \Delta$ and $\mathcal{N}(\sigma) = \mathrm{bc}_{k\uparrow k_d}(\pi)$ for some $\pi \in \mathcal{C}$, or, equivalently, $\tilde{\mathcal{N}}(\sigma)_\phi = \mathrm{bc}_\phi(\pi)$. It suffices to show that:

10.1. Proposition. $\mathrm{bc}_\phi(\pi^\lambda) = \tilde{\mathcal{N}}(\sigma^\lambda)_\phi \in R_\phi(\tilde{\mathcal{N}}(\sigma)_\phi)$ *for all nonzero partitions* λ.

Proof. We consider first the special cases $\lambda = (i)$ and $\lambda = 1^i$:

10.2. Lemma. $\mathrm{bc}_\phi(x_i(\pi)) = \tilde{\mathcal{N}}(x_i(\sigma))_\phi$ *and* $\mathrm{bc}_\phi(y_i(\pi)) = \tilde{\mathcal{N}}(y_i(\sigma))_\phi$.

Proof. The left sides are irreducible in $R_\phi(\tilde{\mathcal{N}}(\sigma)_\phi)$. Since

$$1 = \bigl\langle x_i(\pi), \pi^i\bigr\rangle_{G_{id(\pi)}} = \bigl\langle y_i(\pi), \pi^i\bigr\rangle_{G_{id(\pi)}}$$

and since, by 7.3, bc_ϕ is an isometry it follows that

$$1 = \bigl\langle \mathrm{bc}_\phi\bigl(x_i(\pi)\bigr), \mathrm{bc}_\phi\bigl(\pi^i\bigr)\bigr\rangle_{G_{id(\pi)}(k_d)\phi} = \bigl\langle \mathrm{bc}_\phi\bigl(y_i(\pi)\bigr), \mathrm{bc}_\phi\bigl(\pi^i\bigr)\bigr\rangle_{G_{id(\pi)}(k_d)\phi}.$$

Since $\mathrm{bc}_\phi(\pi^i) = (\mathrm{bc}_\phi(\pi))^i = (\tilde{\mathcal{N}}(\sigma)_\phi)^i$, we see that $\mathrm{bc}_\phi(x_i(\pi))$ and $\mathrm{bc}_\phi(y_i(\pi))$ occur with multiplicity one in $(\tilde{\mathcal{N}}(\sigma)_\phi)^i$. To complete the proof of 10.2 we use:

10.3. Sublemma. $\tilde{\mathcal{N}}(x_i(\sigma))_\phi$ *and* $\tilde{\mathcal{N}}(y_i(\sigma))_\phi$ *are the only irreducible components of* $(\tilde{\mathcal{N}}(\sigma)_\phi)^i$ *which occur with multiplicity one.*

Proof. Let $\theta \in \{x_i(\sigma), y_i(\sigma)\}$. Since $1 = \langle \sigma^i, \theta\rangle_{G_{id(\sigma)}(k_d)}$ and $d(\pi) = l(\sigma)d(\sigma)$, it follows from 8.2 that $1 = \langle \mathcal{N}(\sigma^i), \mathcal{N}(\theta)\rangle_{G_{id(\pi)}(k_d)}$. Since $\mathcal{N}(\sigma^i) = \mathcal{N}(\sigma)^i$ and $(\tilde{\mathcal{N}}(\sigma)_\phi)^i$ is the restriction to $G_{id(\pi)}(k_d)\phi$ of the character $(\tilde{\mathcal{N}}(\sigma))^i := \mathrm{ind}_{\tilde{P}}^{\tilde{G}} \tilde{\mathcal{N}}(\sigma)^{\otimes i}$ ($\tilde{P} := \langle\phi\rangle \ltimes P(k_d)$, where P is a k-parabolic subgroup of $G_{id(\pi)}$), it follows from 6.1(ii) that $\langle \tilde{\mathcal{N}}(\theta)_\phi, (\tilde{\mathcal{N}}(\sigma)_\phi)^i\rangle_{G_{id(\pi)}(k_d)\phi} = \zeta$, where ζ is a root of unity. Moreover, $(\tilde{\mathcal{N}}(\sigma)_\phi)^i$ is a positive element of R_ϕ [5] for all $i \geqslant 1$, hence,

[5] For the notion of positivity used in these arguments see [Zel1, §1.2].

since $\tilde{\mathcal{N}}(\theta)_\phi \in \tilde{\Omega}_\phi$ is also positive, it follows that $\zeta \geqslant 0$, so $\zeta = 1$. Since $\tilde{\mathcal{N}}(\sigma)_\phi$ is the primitive irreducible element of $R(\tilde{\mathcal{N}}(\sigma)_\phi)$, we know from the structure theory of primary PSH algebras (see the symmetric group case in Section 2) that exactly two of the irreducible components of $(\tilde{\mathcal{N}}(\sigma)_\phi)^i$ have inner product 1 with $(\tilde{\mathcal{N}}(\sigma)_\phi)^i$. This proves 10.3. □

We have proved that $\{\mathrm{bc}_\phi(x_i(\pi)), \mathrm{bc}_\phi(y_i(\pi))\} = \{\tilde{\mathcal{N}}(x_i(\sigma))_\phi, \tilde{\mathcal{N}}(y_i(\sigma))_\phi\}$. But it is well known that $\mathrm{bc}_{k\uparrow k_d}$ takes generic characters to generic characters (see, e.g., [SZ2, B3.3(ii)]). This implies 10.2. □

Now let λ be a nonzero partition. The character π^λ is characterized by the fact that it is the only irreducible character which has inner product one with both $x_\lambda(\pi)$ and $y_{\lambda^t}(\pi)$ (see Section 2). Since bc_ϕ is an isometry, $\mathrm{bc}_\phi(\pi^\lambda)$ is also characterized by the property that it has inner product one with $\mathrm{bc}_\phi(x_\lambda(\pi))$ and $\mathrm{bc}_\phi(y_{\lambda^t}(\pi))$.

To prove 10.1 it suffices to prove:

10.4. Lemma.

$$1 = \big\langle \tilde{\mathcal{N}}\big(\sigma^\lambda\big)_\phi, \mathrm{bc}_\phi\big(x_\lambda(\pi)\big)\big\rangle_{G_{|\lambda|d(\pi)}(k_d)\phi} = \big\langle \tilde{\mathcal{N}}\big(\sigma^\lambda\big)_\phi, \mathrm{bc}_\phi\big(y_{\lambda^t}(\pi)\big)\big\rangle_{G_{|\lambda|d(\pi)}(k_d)\phi}.$$

Proof. Since $1 = \langle x_\lambda(\sigma), \sigma^\lambda\rangle_{G_{|\lambda|d(\sigma)}(k_d)}$ and $d(\pi) = l(\sigma)d(\sigma)$, 8.2 implies that

$$1 = \big\langle \mathcal{N}\big(x_\lambda(\sigma)\big), \mathcal{N}\big(\sigma^\lambda\big)\big\rangle_{G_{|\lambda|d(\pi)}(k_d)} = \big\langle \mathcal{N}\big(x_{\ell_1}(\sigma)\big)\cdots\mathcal{N}\big(x_{\ell_r}(\sigma)\big), \mathcal{N}\big(\sigma^\lambda\big)\big\rangle_{G_{|\lambda|d(\pi)}(k_d)}.$$

On the other hand,

$$\begin{aligned}\mathrm{bc}_\phi\big(x_\lambda(\pi)\big) &= \mathrm{bc}_\phi\big(x_{\ell_1}(\pi)\big)\,\mathrm{bc}_\phi\big(x_{\ell_2}(\pi)\big)\cdots\mathrm{bc}_\phi\big(x_{\ell_r}(\pi)\big)\\ &= \tilde{\mathcal{N}}\big(x_{\ell_1}(\sigma)\big)_\phi\tilde{\mathcal{N}}\big(x_{\ell_2}(\sigma)\big)_\phi\cdots\tilde{\mathcal{N}}\big(x_{\ell_r}(\sigma)\big)_\phi,\end{aligned}$$

in which the last equality follows from 10.2 and 7.3. Since $\tilde{\mathcal{N}}(x_i(\sigma))_\phi \in \tilde{\Omega}_\phi$ for all $i \geqslant 1$, $\tilde{\mathcal{N}}(x_{\ell_1}(\sigma))_\phi\tilde{\mathcal{N}}(x_{\ell_2}(\sigma))_\phi\cdots\tilde{\mathcal{N}}(x_{\ell_r}(\sigma))_\phi$ is positive, and since $\tilde{\mathcal{N}}(\sigma^\lambda)_\phi \in \tilde{\Omega}_\phi$ is also positive, $\langle\tilde{\mathcal{N}}(x_{\ell_1}(\sigma))_\phi\cdots\tilde{\mathcal{N}}(x_{\ell_r}(\sigma))_\phi, \tilde{\mathcal{N}}(\sigma^\lambda)_\phi\rangle_{G_{|\lambda|d(\pi)}(k_d)\phi} \geqslant 0$. As in 10.3, we see that $\tilde{\mathcal{N}}(x_{\ell_1}(\sigma))_\phi\cdots\tilde{\mathcal{N}}(x_{\ell_r}(\sigma))_\phi$ is the restriction to $G_{|\lambda|d(\sigma)}(k_d)\phi$ of an extension to $\tilde{G}_{|\lambda|d(\sigma)}$ of $\mathcal{N}(x_\lambda(\sigma))$, so we may apply 6.1(ii) to prove that this inner product is a root of unity. Since the root of unity is positive, it is 1. A similar argument proves the second case of 10.4 and completes the proofs of 10.4 and 10.1. □

Appendix A. The Macdonald correspondence, tempered types, and the level zero local Langlands correspondence

A.0. Introduction and statement of the result

Let K be a p-adic local field with ring of integers O_K, prime ideal $\mathfrak{p}$, and residue class field k. Let $U_K := O_K^\times$ and let $U_K^1 := 1 + \mathfrak{p} \subset U_K$ be the kernel of the reduction map $U_K \longrightarrow k^\times$. Set $G_n := \mathrm{GL}_n$. For $n \geqslant 1$ let $\Omega_n(K)$ denote the set of irreducible smooth representations of $G_n(K)$ and let $\Omega_n(K)_0$ denote the subset of level-zero irreducible representations. Let $\Omega_n(k)$

denote the set of irreducible representations of $G_n(k)$, $\mathcal{C}_n(k) \subset \Omega_n(k)$ the subset consisting of cuspidal representations. Let φ_0 denote the map which assigns each $\Pi \in \Omega_n(K)_0$ the reduction mod $\mathfrak{p}$ of its *tempered type* (Section A.1.3, [SchZ]), let $\mathcal{J}_0$ be the natural map which sends tame n-dimensional representations of the Weil–Deligne group $\mathrm{Rep}_n(W'_K/P)$ to their I-equivalence classes (see Section 4; in particular, (4.1) and (4.2)), let LLC_0 be the restriction to level zero of the local Langlands correspondence, and let $\mathcal{M}_n$ be the Macdonald correspondence (see the end of Section 4).

The purpose of this Addendum is to show that the following diagram is commutative:

$$\begin{array}{ccc} \Omega_n(K)_0 & \xrightarrow{\mathrm{LLC}_0} & \mathrm{Rep}_n(W'_K/P) \\ \varphi_0 \downarrow & & \downarrow \mathcal{J}_0 \\ \Omega_n(k) & \xrightarrow{\mathcal{M}_n} & \mathrm{Rep}_n(W'_K/P)_I. \end{array} \tag{A.1}$$

The work of Macdonald [Mac1] suggests this diagram, although, writing before 1980, Macdonald had neither a theory of tempered types nor the detailed knowledge of the local Langlands correspondence which might permit a definite statement. The paper [SchZ] concerning tempered types allows one to define the mapping φ_0 and thus "close" the diagram; the characterization of level-zero tempered types and the explicit results concerning the local Langlands correspondence of Henniart [He1,He2] make it possible to show that the level-zero diagram (A.1) really is commutative.

We close this Introduction by introducing some notation and quickly summarizing the organization of this Addendum.

Let $\Omega(K) = \bigsqcup_{n \geqslant 1} \Omega_n(K)$ and let $\mathcal{C}(K) \subset \Omega(K)$ be a set of unitary representatives for the set of all unramified twist classes of irreducible cuspidal representations of $G_n(K)$ for all n. For $\Sigma \in \mathcal{C}(K)$ let $[\Sigma]$ be the set of all unramified twists of Σ. Thus the disjoint union $\bigsqcup_{\Sigma \in \mathcal{C}(K)} [\Sigma] \subset \Omega(K)$ is the set of all irreducible cuspidal representations of $G_n(K)$ for all $n \geqslant 1$.

In Section A.1 we make the tempered type mapping φ_0 explicit. In rough terms, φ_0 sends $\Pi \in \Omega_n(K)_0$ to $\bar{\sigma}_\Pi \in \Omega_n(k)$ and factors as follows. In Section A.1.1 we represent Π by a multisegment $\Delta(\Pi)$ and define from the multisegment a partition-valued function $\Lambda = \Lambda_\Pi$ which is supported on $\mathcal{C}(K)_0 := \mathcal{C}(K) \cap \Omega(K)_0$. In Section A.1.3 we interpret Λ as a partition-valued function on $\mathcal{C}(k)$ and convert Λ back into a character $\bar{\sigma}_\Pi \in \Omega_n(k)$ (see 3.1 and (A.4)). The tempered type of Π is the inflation of $\bar{\sigma}_\Pi$ to $G_n(O_K)$. In Section A.1.2 we recall Bushnell–Kutzko types. Section A.1 concludes with some additional properties of tempered types, including their connection to Bushnell–Kutzko types.

In Section A.2 we show that the mapping φ_0 is indeed the mapping which makes the diagram (A.1) commutative. We conclude the Addendum by pointing out that the mapping φ_0 connects the finite field and local field base-change maps (see (A.7)).

A.1. Tempered types and the mapping φ_0

A.1.1. Multisegments and partition-valued functions

The Langlands classification theorems imply that every element $\Pi \in \Omega_n(K)$ has a unique realization as a "Langlands quotient": For every Π there is a unique standard parabolic subgroup $P(K) = L(K) \ltimes U_P(K) \subset G_n(K)$, a unique tempered representation π of the Levi factor $L(K)$, and a unique regular, positive[6] unramified twist $\chi_\nu \otimes \pi$ such that $\Pi = j_{G_n(K),P(K)}(\chi_\nu \otimes \pi)$, where $j_{G_n(K),P(K)}(\chi_\nu \otimes \pi)$ denotes the unique irreducible quotient representation of the normalized parabolic induction $i_{G_n(K),P(K)}(\chi_\nu \otimes \pi)$. Since every irreducible tempered representation of $G_n(K)$ is the irreducible normalized parabolic induction of some discrete series representation of a Levi subgroup, it follows that, in the case of $G_n(K)$, it is possible to drop the "regularity" condition on the twist and assume that π is a discrete series representation of $L(K)$. Thus, Zelevinsky, in classifying the elements of $\Omega_n(K)$, realizes $\Pi \in \Omega_n(K)$ as a Langlands quotient

$$\Pi = j(\pi_1 \times \cdots \times \pi_r) \tag{A.2}$$

of a Hopf product, where each π_i is an irreducible essentially discrete series representation of some $G_{n_i}(K)$ and where the multiset $\{\pi_1, \dots, \pi_r\}$ is uniquely determined by Π. According to Bernstein–Zelevinsky [Zel2, 9.3], the cuspidal support of each π_i is a "segment"; more precisely, the segment $\Delta(\pi_i)$ associated to π_i consists of ℓ_{π_i} unramified twists of a unique element $\Sigma_i \in \mathcal{C}(K)$.

Zelevinsky's classification of $\Omega_n(K)$ naturally assigns a "multisegment" to $\Pi \in \Omega_n(K)$[7]:

$$\Pi \longrightarrow \Delta(\Pi) := \big\{\Delta(\pi_1), \dots, \Delta(\pi_r)\big\}.$$

Let $\mathcal{P}(\mathcal{C}(K))$ denote the set of all finitely-supported, partition-valued functions on $\mathcal{C}(K)$. To Π or $\Delta(\Pi)$ we assign a function $\Lambda_\Pi \in \mathcal{P}(\mathcal{C}(K))$ as follows: For $\Sigma \in \mathcal{C}(K)$ let $\pi_{i_1}, \dots, \pi_{i_s}$ be the set of all the π_i from (A.2) which have cuspidal support in $[\Sigma]$. Then $\Lambda_\Pi(\Sigma) := (\ell_1, \dots, \ell_s)$ is the partition such that ℓ_j is the length of the segment $\Delta(\pi_{i_j})$ for $1 \leqslant j \leqslant s$.[8]

A.1.2. The Bernstein spectrum and Bushnell–Kutzko types

Before explaining the mapping from Λ_Π to the tempered type of Π in the level-zero case we want to explain how Bushnell–Kutzko types fit into the picture. Let $\Lambda \in \mathcal{P}(\mathcal{C}(K))$. Then the *cuspidal divisor of* Λ,

$$D(\Lambda) := \sum_{\Sigma \in \mathcal{C}(K)} \big|\Lambda(\Sigma)\big| \Sigma \in \mathrm{Div}^+\big(\mathcal{C}(K)\big), \tag{A.3}$$

determines a connected component $\mathrm{Conn}_{D(\Lambda)}$ of the Bernstein spectrum of $G_n(K)$, where $n = |D(\Lambda)|$ is the *degree of* $D(\Lambda)$.

For every $D \in \mathrm{Div}^+(\mathcal{C}(K))$, a *type* for Conn_D is a pair (J, μ) consisting of an open compact subgroup J, which we may assume to be contained in $G_n(O_K)$, and an irreducible representation μ of J such that, if $\Pi \in \Omega_n(K)$, then $\mu \subset \Pi|_J$ if and only if the cuspidal support of Π is in Conn_D. For arbitrary reductive groups and connected components of their Bernstein spectrum types, if they exist at all, are not unique [BK2]. However, for $G_n(K)$ Bushnell and Kutzko [BK1, BK3] have constructed a certain family of types such that each Bernstein component has a type from that family which is unique up to conjugacy. We will call these types *Bushnell–Kutzko types*. Here we will be concerned only with level-zero types, and we note that [GSZ], which deals with simple algebras, also determines the level-zero Bushnell–Kutzko types for $G_n(K)$.

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[6] The word *positive* has a double meaning: First, the quasicharacter χ_ν has only positive real values and, second, the parameter ν lies in a positive chamber relative to P.

[7] A twist of a multisegment is a multisegment.

[8] In [SchZ, §2] the notation was $\mathcal{P}$ instead of Λ_Π, and the relation of Π to $\mathcal{P}$ was expressed as $\Pi \in \mathrm{im}(Q_{\mathcal{P}})$.

Fact 1. *For $\Sigma \in \mathcal{C}(K)_0$ the Bushnell–Kutzko type of the Bernstein component $[\Sigma]$ is the pair $(G_n(O_K), \sigma(\Sigma))$ such that, for all $\Pi \in [\Sigma]$, $\Pi = \mathrm{Ind}_{G_n(O_K)K^\times}^{G_n(K)}(\sigma \otimes \eta(\Pi))$, where $\eta(\Pi)$ is the central character of Π. Moreover, $\sigma(\Sigma)$ is also the tempered type of all $\Pi \in [\Sigma]$.*

Proof. Since $\sigma(\Sigma) \subset \Pi|_{G_n(O_K)}$ for all $\Pi \in \Sigma$, it is clear that $(J, \mu) = (G_n(O_K), \sigma(\Sigma))$ is a type for the Bernstein component $[\Sigma]$. The tempered type σ_Λ is always an irreducible representation of $G_n(O_K)$ and it is, in addition, always true that the Bushnell–Kutzko type of $\mathrm{Conn}_{D(\Lambda)}$ satisfies $\mu \subset \sigma_\Lambda|_J$. Therefore, in this case the tempered type satisfies $\sigma_{\Lambda_\Pi} = \sigma(\Sigma)$ for all $\Pi \in [\Sigma]$. □

A.1.3. The discrete series spectrum of $G_n(K)$ and tempered types

Following [SchZ] we now make the mapping φ_0 explicit. Recall that φ_0 assigns $\Pi \in \Omega(K)_0$ the reduction $\bar{\sigma}_\Lambda$ of the tempered type σ_Λ, where $\Lambda = \Lambda_\Pi$ (Section A1.1). Each level-zero $\Sigma \in \mathcal{C}_s(K)_0$ ($s \geqslant 1$) has, by Fact 1, a unique Bushnell–Kutzko type $\sigma(\Sigma)$, which is an irreducible level-zero representation of $G_s(O_K)$.

Fact 2. *If $\Pi \in \Omega(K)_0$, then $\varphi_0(\Pi) = \bar{\sigma}_{\Lambda_\Pi}$ is the following Hopf product on $G_n(k)$:*

$$\varphi_0(\Pi) = \bar{\sigma}_{\Lambda_\Pi} = \prod_{\Sigma \in \mathcal{C}(K)_0} \bigl(\bar{\sigma}(\Sigma)^{\Lambda_\Pi(\Sigma)}\bigr). \tag{A.4}$$

The tempered type σ_{Λ_Π} of Π is the inflation of $\bar{\sigma}_{\Lambda_\Pi}$ to $G_n(O_K)$.

Proof. We have seen in Fact 1 that, if $\Pi \in [\Sigma] \subset \mathcal{C}(K)_0$, the Bushnell–Kutzko and tempered types are the same irreducible representation $\sigma(\Sigma)$ of $G_n(O_K)$ such that $\bar{\sigma}(\Sigma) \in \mathcal{C}(k)$. Therefore, for $\Pi \in [\Sigma]$ a cuspidal representation, $\varphi_0(\Pi) = \bar{\sigma}(\Sigma)$.

If the cuspidal divisor $D(\Lambda_\Pi)$ is simple, i.e. if

$$D(\Lambda_\Pi) = \bigl|\Lambda_\Pi(\Sigma)\bigr|\Sigma = r\Sigma \quad \bigl(r := \bigl|\Lambda_\Pi(\Sigma)\bigr|\bigr),$$

then the Bushnell–Kutzko type of $\mathrm{Conn}_{D(\Lambda_\Pi)}$ is $(J, \mu) = (\mathfrak{A}_r^\times, (\bar{\sigma}(\Sigma)^{\otimes r}) \circ p)$, where $\mathfrak{A}_r^\times \subset G_n(O_K)$ denotes the unit group of the standard principal order of period r ($rs = n$) and p is the projection map $p : \mathfrak{A}_r^\times \longrightarrow G_s^r(k)$.

In the notation of [SchZ], $J_{\max} = G_n(O_K)$ and $\kappa_{\max}$ is the trivial representation. The defining equation on the last line of [SchZ, p. 185] means that the reduction of the tempered type σ_{Λ_Π} is $\bar{\sigma}(\Sigma)^{\Lambda_\Pi(\Sigma)}$.

Now we consider Π semisimple, i.e. the support of the cuspidal divisor of Λ_Π consists of more than one element of $\mathcal{C}(K)_0$. A partition-valued function $\Lambda \in \mathcal{P}(\mathcal{C}(K))$ may be regarded as a tuple of functions Λ_Σ ($\Sigma \in \mathcal{C}(K)$), each defined such that $\Lambda_\Sigma(\Sigma) = \Lambda(\Sigma)$ and $\Lambda_\Sigma(\Sigma') = 0$ for $\Sigma' \neq \Sigma$. In [SchZ, §6 definition p. 188 above Prop. 2] the tempered type σ_Λ is constructed from the tempered types σ_{Λ_Σ} which are associated to simple Bushnell–Kutzko types. In the level-zero case $\Lambda \in \mathcal{P}(\mathcal{C}(K)_0)$ we may consider σ_Λ as a representation of $G_n(O_K)$ or, by reduction, as a representation $\bar{\sigma}_\Lambda$ of $G_n(k)$. In the $G_n(k)$ setting the definition means that $\bar{\sigma}_\Lambda$ is the Hopf product of the simple tempered types $\bar{\sigma}_{\Lambda_\Sigma}$. Thus, the proof of the assertion concerning the general $\bar{\sigma}_\Lambda$ reduces to the simple case, which was proved above. □

Remarks.

(i) The Bushnell–Kutzko type of $\Pi \in \Omega_n(K)_0$ has a natural reduction which can be identified with the cuspidal support of $\varphi_0(\Pi)$.
(ii) If $\Pi = \Delta(\Pi)$ is an essentially discrete series representation, i.e. a segment of length ℓ with cuspidal support in $[\Sigma] \subset \mathcal{C}(K)_0$, then $\varphi_0(\Pi) = x_\ell(\bar{\sigma}(\Sigma))$ (see Section 2).

In general, whether level-zero or not, up to conjugacy, σ_Λ is an irreducible representation of the maximal compact subgroup $G_n(O_K)$ of $G_n(K)$. The mapping $\Pi \longmapsto \sigma_\Lambda$ factors through $\mathcal{P}(\mathcal{C}(K))$, which serves as a parameter set for the set of (conjugacy classes of) tempered types as well as for the set of connected components of the tempered spectrum of $G_n(K)$ for all $n \geqslant 1$.

Conversely, up to conjugacy in $G_n(K)$, each $\Lambda \in \mathcal{P}(\mathcal{C}(K))$ of degree

$$\sum_{\Sigma \in \mathcal{C}(K)} \bigl|\Lambda(\Sigma)\bigr| |\Sigma| = n$$

determines a discrete series representation of a Levi subgroup $L_\Lambda(K)$,

$$\pi_\Lambda = \bigotimes_{(\Sigma,\ell):\ \ell \in \Lambda(\Sigma)} \pi(\Sigma, \ell)$$

such that $\pi(\Sigma, \ell)$ is the (unique) discrete series representation with the property that $\Delta(\pi(\Sigma, \ell))$ is the *centered segment* consisting of ℓ positive real unramified twists of Σ. Thus Λ also specifies the connected component Θ_Λ of the *tempered spectrum* of $G_n(K)$ which consists of all unitary unramified twists of the conjugacy class of $(L_\Lambda(K), \pi_\Lambda)$. By construction the discrete series support of a tempered representation Π is in Θ_{Λ_Π}. Furthermore, the cuspidal support map injects Θ_Λ into the Bernstein component $\mathrm{Conn}_{D(\Lambda)}$ (see (A.3)).

The tempered type σ_Λ, which corresponds to Λ and therefore to every $\Pi \in \Omega_n(K)$ such that $\Lambda_\Pi = \Lambda$, also corresponds to the Bernstein component $\mathrm{Conn}_{D(\Lambda)}$ in the sense that $\mu \subset \sigma_\Lambda|_J$, where we assume that the Bushnell–Kutzko type (J, μ) of $\mathrm{Conn}_{D(\Lambda)}$ is chosen such that $J \subset G_n(O_K)$. According to [SchZ, Section 6, Proposition 2 and prior remarks]:

Fact 3. *Let (J_D, μ_D) denote the Bushnell–Kutzko type for the Bernstein component* Conn_D. *Then the set of all tempered types $\sigma = \sigma_\Lambda$ such that $D(\Lambda) = D$ consists of all irreducible representations σ of $G_n(O_K)$ such that $\mu_D \subseteq \sigma|_{J_D}$.*

Fact 4. *Fix $\Pi \in \Omega_n(K)$, let $\Lambda = \Lambda_\Pi \in \mathcal{P}(\mathcal{C}(K))$, and let σ_Λ denote the tempered type corresponding to Λ. Then*:

(i) *$\sigma_\Lambda \subset \Pi|_{G_n(O_K)}$ with multiplicity one.*
(ii) *If $\sigma_{\Lambda'} \subset \Pi|_{G_n(O_K)}$, then $\Lambda' \leqslant \Lambda$, i.e. $\Lambda(\Sigma) \leqslant \Lambda'(\Sigma)$ for all $\Sigma \in \mathcal{C}(K)$ with respect to the natural or dominance partial order of partitions.*
(iii) *If Π is tempered and $\Lambda' \leqslant \Lambda$, then, conversely, $\sigma_{\Lambda'} \subset \Pi|_{G_n(O_K)}$.*

Remark. As noted in Fact 4(ii), (iii), a tempered type is not a type in the sense of Section A.1.2 but in the more general sense of [BK2, §4], where the Hecke algebra has to be replaced by the Schwartz algebra.

A.2. The proof of the commutativity of the diagram (A.1)

We shall show that the tempered type mapping $\varphi_0 : \Omega(K)_0 \longrightarrow \Omega(k)$ (see (A.4)) makes (A.1) into a commutative diagram. We shall do this in two steps, first the case in which $\Pi \in \Omega(K)_0$ is cuspidal and then the general case.

We first note that for any $\Pi \in \Omega(K)_0$ the support of the partition-valued function Λ_Π lies in $\mathcal{C}(K)_0$. Since we have considered a set of representatives $\mathcal{C}(K)$ modulo unramified twist, we have a natural bijection

$$\varphi_0|_{\mathcal{C}(K_0)} : \mathcal{C}(K)_0 \xrightarrow{\sim} \mathcal{C} := \mathcal{C}(k), \qquad \Sigma \longmapsto \bar\sigma = \bar\sigma(\Sigma), \tag{A.5}$$

which assigns the reduction $\bar\sigma \in \mathcal{C}$ of its level-zero Bushnell–Kutzko type $(G_n(O_K), \sigma)$ to a cuspidal level-zero representation Σ. If V_Σ is the representation space and $1 + \mathfrak{P} \subset G_n(O_K)$ the main congruence subgroup, then $\bar\sigma$ acts in $V_\Sigma^{1+\mathfrak{P}}$, which is regarded as a representation of $G_n(k) = G_n(O_K)/1 + \mathfrak{P}$. (See for instance [GSZ, Theorem 5.5(ii)] specialized to the split case.) It follows as in Fact 1 that $\varphi_0(\Pi) = \bar\sigma = \bar\sigma(\Sigma)$ for all $\Pi \in [\Sigma]$, $\Sigma \in \mathcal{C}(K)_0$.

Proposition 1. *If Π is an arbitrary irreducible level-zero cuspidal representation of $G_n(K)$, i.e. if $\Sigma \in \mathcal{C}(K)_0$ and $\Pi \in [\Sigma]$, then $\mathcal{M} \circ \varphi_0(\Pi) = \mathcal{J}_0 \circ \mathrm{LLC}(\Pi)$.*

Proof. Since $\varphi_0(\Pi) = \varphi_0(\Sigma)$ and $\mathcal{J}_0 \circ \mathrm{LLC}(\Pi) = \mathrm{LLC}(\Pi)_I = \mathrm{LLC}(\Sigma)_I$, we have to prove only that

$$\mathcal{M} \circ \varphi_0(\Sigma) = \mathrm{LLC}(\Sigma)_I.$$

Since Σ is cuspidal and level-zero, the representation $\mathrm{LLC}(\Sigma)$ of W'_K/P is irreducible, so it may be regarded as an irreducible representation of W_K/P (see Section 4). Thus $\mathrm{LLC}(\Sigma) = \rho(\chi_n) = \mathrm{Ind}_{K_n}^K(\chi_n)$, where $K_n|K$ is the unramified extension of degree n for some integer $n \geqslant 1$ and χ_n is a K-regular character of $K_n^\times / U_{K_n}^1$. In this case, $\mathrm{LLC}(\Sigma)_I$ is, by definition, the set of all representations of W_K/P which have the same restriction to I as $\rho(\chi_n)$. As in Section 4 we identify $X(I/P)$ with $X(\varprojlim_N k_d^\times)$ and the restriction $\chi_n|_I$ with the reduction $\bar\chi_n$ of χ_n, which is a k-regular character in $X(k_n^\times)$ (see 3.1(ii)). We identify the I-equivalence class of $\mathrm{LLC}(\Sigma)$ with the support of its restriction to I, which is the Galois orbit $[\bar\chi_n]$ of $\bar\chi_n$. Thus our displayed equation becomes

$$\mathcal{M} \circ \varphi_0(\Sigma) = [\bar\chi_n].$$

But $\Sigma = \mathrm{LLC}^{-1}(\rho(\chi_n)) = \mathcal{A}_{K_n|K}(\chi_n)$ is the automorphic induction of χ_n. In fact we are here in the easiest case of automorphic induction, first studied by David Kazhdan. Inverting $\mathcal{M}$ we must still show that

$$\varphi_0\big(\mathcal{A}_{K_n|K}(\chi_n)\big) = \mathcal{M}^{-1}\big([\bar\chi_n]\big).$$

The right side is the cuspidal representation $\pi(\bar\chi_n) \in \mathcal{C}$ with the Green's parameter $[\bar\chi_n]$ (cf. Section 4). Therefore, it suffices to show that the automorphic induction $\mathcal{A}_{K_n|K}(\chi_n)$ has the inflation of $\pi(\bar\chi_n)$ as its level zero type. But this fact is a special case of the main result of [He1], which compares $\mathcal{A}_{K_n|K}(\chi_n)$ with a representation $\Pi'(\chi_n)$ which is constructed in an explicit way using $\pi(\bar{\chi}_n)$ and shows that the two representations differ at most by an unramified twist. $\square$

Now we want to show that the mapping $\varphi_0 : \Omega(K)_0 \longrightarrow \Omega(k)$, as defined in (A.4), makes (A.1) into a commutative diagram.

Proposition 2. $\mathcal{M} \circ \varphi_0(\Pi) = \mathcal{J}_0 \circ \mathrm{LLC}(\Pi)$ *for all* $\Pi \in \Omega_n(K)_0$.

Proof. Let Π be represented as the Langlands quotient $j(\pi_1 \times \cdots \times \pi_r)$. Then, according to [He2, 2.9],

$$\mathrm{LLC}\big(j(\pi_1 \times \cdots \times \pi_r)\big) = \mathrm{LLC}(\pi_1) \oplus \cdots \oplus \mathrm{LLC}(\pi_r). \tag{A.6}$$

If $\Lambda_\Pi(\Sigma) = (\ell_1, \ldots, \ell_s)$, where ℓ_j is the length of $\Delta(\pi_{i_j})$ (Section A.1.1), then [He2, 2.7] implies that $\mathrm{LLC}(\pi_{i_j})$ and $\mathrm{sp}(\ell_j) \otimes \mathrm{LLC}(\Sigma)$ differ only by an unramified twist. It follows that, up to unramified twists, we may rewrite (A.6) as

$$\mathrm{LLC}\big(j(\pi_1 \times \cdots \times \pi_r)\big) = \bigoplus_{\Sigma \in \mathcal{C}(K)_0} \mathrm{sp}\big(\Lambda(\Sigma)\big) \otimes \mathrm{LLC}(\Sigma),$$

where $\Lambda = \Lambda_\Pi$ and $\mathrm{sp}(\Lambda)$ has the same meaning as in Section 4. On the other hand, the Macdonald correspondence gives:

$$\mathcal{M}(\bar{\sigma}_\Lambda) = \bigoplus_{\Sigma \in \mathcal{C}(K)_0} \mathcal{M}\big(\bar{\sigma}(\Sigma)^{\Lambda(\Sigma)}\big) = \bigoplus_{\Sigma \in \mathcal{C}(K)_0} \mathrm{sp}\big(\Lambda(\Sigma)\big) \otimes \mathcal{M}\big(\bar{\sigma}(\Sigma)\big).$$

Therefore the assertion that $\mathrm{LLC}(j(\pi_1 \times \cdots \times \pi_r))_I = \mathcal{M}(\bar{\sigma}_\Lambda)$ reduces to the assertion that $\mathrm{LLC}(\Sigma)_I = \mathcal{M}(\varphi_0(\Sigma)) = \mathcal{M}(\bar{\sigma}(\Sigma))$, which was proved in Proposition 1. $\square$

Final Remark. Let us conclude by showing that the mapping to the tempered type $\varphi_0 : \Omega(K)_0 \longrightarrow \Omega(k)$ is compatible with both the local field and finite field base change maps, i.e. that

$$\varphi_{0,k_d} \circ \mathrm{bc}_{K\uparrow K_d} = \mathrm{bc}_{k\uparrow k_d} \circ \varphi_{0,k}. \tag{A.7}$$

The essential point is that the main result of the main part of our paper (1.1),

$$\mathcal{M}_{k_d} \circ bc_{k\uparrow k_d} = \mathrm{res}_{K\downarrow K_d} \circ \mathcal{M}_k, \tag{A.8}$$

as well as the local field relation,

$$\mathrm{LLC}_{K_d} \circ \mathrm{bc}_{K\uparrow K_d} = \mathrm{res}_{K\downarrow K_d} \circ \mathrm{LLC}_K, \tag{A.9}$$

proved by Henniart (see [He3, §7]) by global arguments established in [AC], provide examples of “Langlands functoriality.”[9] From (A.9) we see that $\mathrm{bc}_{K\uparrow K_d}$ preserves level zero, so we have a map

[9] Our ostensibly local proof of (A.8) was based on the cuspidal case, which is proved in [SZ2] (the proof depends upon AMT, so it too involves global arguments), or it is proved in [Gy,Di] by using results of Deligne–Lusztig.

$$\mathrm{bc}_{K\uparrow K_d} : \Omega(K)_0 \longrightarrow \Omega(K_d)_0^{\phi}.$$

Since $\mathcal{J}_0 \circ \mathrm{res}_{K\downarrow K_d} = \mathrm{res}_{K\downarrow K_d} \circ \mathcal{J}_0$ and since "restriction" preserves I-equivalence classes, we obtain:

$$\begin{aligned}
\mathcal{M}_{k_d} \circ \varphi_{0,k_d} \circ bc_{K\uparrow K_d} &\underset{(\mathrm{a}.1)}{=} \mathcal{J}_0 \circ \mathrm{LLC}_{K_d,0} \circ \mathrm{bc}_{K\uparrow K_d} \\
&\underset{(\mathrm{a}.9)}{=} \mathcal{J}_0 \circ \mathrm{res}_{K\downarrow K_d} \circ \mathrm{LLC}_{K,0} \\
&= \mathrm{res}_{K\downarrow K_d} \circ \mathcal{J}_0 \circ \mathrm{LLC}_{K,0} \\
&\underset{(\mathrm{a}.1)}{=} \mathrm{res}_{K\downarrow K_d} \circ \mathcal{M}_k \circ \varphi_{0,k} \\
&\underset{(\mathrm{a}.8)}{=} \mathcal{M}_{k_d} \circ bc_{k\uparrow k_d} \circ \varphi_{0,k}.
\end{aligned}$$

This proves (A.7).

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