Langlands classification for L-parameters

A talk dedicated to Sergei Vladimirovich Vostokov on the occasion of his 70th birthday
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In the representation theory of a connected reductive group G with points in a local field the Langlands classification theorems reduce the problem of classifying all irreducible representations of G to that of classifying the tempered representations for G and its standard Levi subgroups. The tempered representation of a Levi subgroup associated to a given representation π of G may be called the tempered support of π.

The background for our considerations is the local Langlands conjecture which predicts a natural parametrization of the (L-packets of) irreducible representations of G in terms of certain Galois parameters introduced by Langlands; we call them the L-parameters. An L-parameter for G is basically an equivalence class $[\phi]$ of Galois representations with values in a complex reductive group $\widehat{G}$ which in a sense is dual to G. In particular, one expects that the L-parameters for tempered representations will satisfy a certain boundedness condition, and we will speak of a tempered L-parameter if that condition holds.

In this talk we address the parallel problem of classifying the L-parameters for G and reduce it to classifying the tempered L-parameters for G and for its standard Levi subgroups. Just as in representation theory, every L-parameter $[\phi]$ for G has its tempered support which is a tempered L-parameter for a certain standard Levi subgroup of G. This suggests a reduction of the local Langlands conjecture to the problem of matching tempered L-packets to tempered L-parameters.
Preliminary results pointing into that direction we have found in **[A]** and **[H]**.

1. Introduction, Langlands classification for representations

Let $G = G(F)$ be a connected reductive group with coefficients in a p-adic field F. One of the main problems of the local Langlands program is **to classify the set** $Irr(G)$ of (equivalence classes of) irreducible admissible complex representations. So we have G acting on some $V|\mathbb{C}$, and the action is said to be **smooth** if for all vectors the stabilizer $Stab_G(v) \subset G$ is an open subgroup, and it is said to be **admissible** if moreover the spaces V^U of fixed vectors for open subgroups $U \subseteq G$ are always of finite dimension.
The Langlands classification is a reduction of that problem to classify the tempered representations of G and of its standard Levi subgroups.
The (irreducible) tempered representations of G sit between unitary and square integrable representations:

$$\{\text{unitary reps}\} \supset \{\text{tempered reps}\} \supset \{\text{square integrable reps}\}.$$

Basically this means that matrix coefficients (=functions of the form $g \in G \mapsto \langle gv, \widetilde{v}\rangle \in \mathbb{C}$ where $\widetilde{v}$ denotes a linear form on V) are square integrable up to an ϵ (i.e. replace $|.|^2$ by

$|.|^{2+\epsilon}$).

Prerequisites: In G we fix $A_0,\ M_0,\ P_0$ a maximal F-split torus, its centralizer, and an F-parabolic subgroup which admits M_0 as a Levi subgroup. The relative Weyl-group ${}_FW = W(A_0, G)$ acts on $\mathfrak{a}^*_{M_0} := \mathbb{R} \otimes X^*(M_0)_F$, and we fix a euclidean structure $< .,. >$ such that the action of ${}_FW$ becomes orthogonal.

Based on that we have the notion of **standard triple** (P, σ, ν) consisting of

(i) a standard F-parabolic subgroup $P \supseteq P_0$,

(ii) an irreducible tempered representation σ of the standard Levi subgroup $M = M_P$,

(iii) an element $\nu \in \mathfrak{a}^*_M \overset{res}{\hookrightarrow} \mathfrak{a}^*_{M_0}$ which is **regular with respect to** P, i.e. it is properly contained in the conic chamber of $\mathfrak{a}^*_M$ which is determined by the roots of A_M(=maximal F-split torus in the center of M) acting on $Lie(N_P)$.

We recall that $\nu \in \mathfrak{a}^*_M = \mathbb{R} \otimes X^*(M)_F$ determines a positive real valued unramified character χ_ν of $M = M(F)$ in such a way that a pure tensor $\nu = s \otimes \theta$ is sent to the character $\chi_\nu(m) = |\theta(m)|^s_F$. It is well defined because θ should be F-rational, $\theta : M = M(F) \to F^\times$. Then we have a well defined bijection

$$\{(P, \sigma, \nu) \mid \text{standard triples}\} \longleftrightarrow Irr(G)$$

sending (P, σ, ν) to

$$\pi(P, \sigma, \nu) := j(i_{G,P}(\sigma \otimes \chi_\nu)),$$

where $i_{G,P}$ is the normalized parabolic induction, and j is its uniquely determined irreducible quotient (quotient theorem of Langlands for real groups and of Silberger for p-adic groups). Of course this is only a reduction to tempered representations; if π is tempered by itself then the corresponding standard triple $(G, \pi, 0)$ carries no further information.

On the other hand Langlands proposed to consider $\Phi(G)$ the set of L-parameters for G such that

$$Irr(G) = \bigcup_{\phi \in \Phi(G)} \Pi_\phi$$

should be the disjoint union of L-packets Π_ϕ consisting of all irreducible representations with Langlands parameter ϕ.

The aim of this talk is to give a Langlands classification of the set $\Phi(G)$ of L-parameters:

$$\Phi(G) \longleftrightarrow \{(P, {}^t\phi, \nu) \mid \text{L-parameter standard triples}\}$$

where now ${}^t\phi$ is a tempered L-parameter of M_P. This suggests:

If $\phi \leftrightarrow (P, {}^t\phi, \nu)$, then the L-packet Π_ϕ should consist of representations $\pi \leftrightarrow (P, \sigma, \nu)$ where the data (P, ν) are fixed by ϕ and where σ is running over the tempered L-packet $\Pi^M_{{}^t\phi}$ for the group $M = M_P$.

Appendix: For $G = GL_n$ Bernstein and Zelevinsky could even reduce the classification of $Irr(G)$ to classifying the square integrable representations. Let σ be a square integrable representation of the standard Levi-group M and let $M' \supset M$ be a larger standard group. The corresponding standard parabolic groups are $P = M \cdot P_0$ and $P' = M' \cdot P_0$ resp. The normalized parabolic induction $i_{M',P \cap M'}(\sigma)$ will be unitary hence it will be the direct sum of irreducible unitary representations **which are all tempered.** Actually this is true for all G but for $G = GL_n$ the induction of a square integrable σ will be **again irreducible.** This yields a classification of $Irr(GL_n)$ in terms of modified triples (P, σ, ν) where now σ is square integrable and ν can also be on the boundary of the corresponding P-chamber in $\mathfrak{a}_M^*$. We note that the boundary of the chamber consists of faces of different dimension, each face related to a parabolic subgroup larger than P. Now we take the face of minimal dimension which contains ν, i.e. relative to that face the ν will be regular. Let $P',\ M'$ be the corresponding standard groups. Then we obtain

$$i_{M',P\cap M'}(\sigma \otimes \chi_\nu) = i_{M',P\cap M'}(\sigma) \otimes \chi_\nu,$$

and the triple (P', σ', ν) where $\sigma' = i_{M',P\cap M'}(\sigma)$, is a standard triple in the original sense.

2. Langlands' dual group and the notion of L-parameters

Let $G|\overline{F}$ be a connected reductive group over a separably closed field and fix $B \supset T$ a Borel subgroup and a maximal torus in G. Then assigned to (G, B, T) is a based root datum

$$\psi_0(G, B, T) = \{X^*(T), \Delta^*(T, B), X_*(T), \Delta_*(T, B)\},$$

consisting of the dual lattices of rational characters and cocharacters and of the corresponding sets of simple roots and coroots resp. The perfect duality is invariant under the action of the Weyl group $W(T, G)$.

We will not go into the notion of an abstract based root datum, but interchanging the role of * and $_*$ preserves the axioms and produces again a based root datum $\widehat{\psi}_0(G, B, T)$. Then from the theory of algebraic groups we know that there is (unique up to isomorphism) a complex reductive group $(\widehat{G}, \widehat{B}, \widehat{T})|\mathbb{C}$ such that

$$\psi_0(\widehat{G}, \widehat{B}, \widehat{T}) = \widehat{\psi}_0(G, B, T).$$

Going from G to $\widehat{G}$ will preserve the root systems A, D, whereas the root systems B, C will be interchanged.

Now assume $G = G(F)$ and $\Gamma := Gal(\overline{F}|F)$ the Galoisgroup of a separable closure of F. Then we may define a rational Galois action on the geometric objects (G, B, T) and $(\widehat{G}, \widehat{B}, \widehat{T})$ as follows:

$$\begin{array}{ccccc} \Gamma & \xrightarrow{\mu} & Aut(\psi_0) & \hookrightarrow & Aut(G, B, T)\ . \\ & & \big\downarrow \cong & & \\ & & Aut(\widehat{\psi}_0) & \hookrightarrow & Aut(\widehat{G}, \widehat{B}, \widehat{T}) \end{array}$$

Here the map $\mu : \Gamma \to Aut(\psi_0)$ is defined via Galois action on the coefficients of the rational characters $\chi \in X^*(T)$ and cocharacters $\chi^\vee \in X_*(T)$. If (G, B, T) is not quasisplit then this action may also change the maximal torus and has to be conjugated back. The injection into $Aut(G, B, T)$, $Aut(\widehat{G}, \widehat{B}, \widehat{T})$ is also not unique but depends on fixing

generators for the root subgroups G_α and $\widehat{G}_{\alpha^\vee}$ for the simple roots.
The L-group of $G|F$ is now defined as the nonconnected complex reductive group

$$^LG := \widehat{G} \rtimes_\mu \Gamma,$$

or its Weil form which is obtained by replacing the profinite Γ with the dense subgroup $W_F \subset \Gamma$ where only integral powers of Frobenius elements are allowed.

The easiest example is to take $G = T$ an F-torus. Then we obtain $T|F \mapsto \widehat{T}|\mathbb{C}$ from the observation that quite general: $\widehat{T} \cong \mathbb{C}^\times \otimes X_*(\widehat{T})$. Now the requirement $X_*(\widehat{T}) = X^*(T)$ leads to the definition

$$\widehat{T} := \mathbb{C}^\times \otimes X^*(T), \qquad {}^LT = \widehat{T} \rtimes_\mu W_F,$$

where the μ-action is defined via the rational Galois action on $X^*(T)$.

Some relations $G \quad - \quad \widehat{G} \quad - \quad {}^LG$:
The group $G = G(\overline{F})$ is a group with BN-pair $(G, B, N_G(T), \Delta^*(T, B))$ and $W(T, G)$ is the corresponding Weyl-group. The same is true if we go from (G, B, T) to $(\widehat{G}, \widehat{B}, \widehat{T})$. But moreover the Weyl groups $W(T, G) \cong W(\widehat{T}, \widehat{G})$ are isomorphic including a bijection of generating reflections which comes from $\Delta^*(T, B) \leftrightarrow \Delta_*(T, B) = \Delta^*(\widehat{T}, \widehat{B})$. Therefore in the corresponding simplicial buildings $\mathcal{B}(G)$ and $\mathcal{B}(\widehat{G})$ consisting of parabolic subgroups, the T-apartment and the $\widehat{T}$-apartment can be identified because they are Coxeter complexes for $W(T, G) \cong W(\widehat{T}, \widehat{G})$. So we get a bijection

$$T \subset P,\ M \leftrightarrow \widehat{P},\ \widehat{M} \supset \widehat{T},$$

between parabolic and Levi subgroups containing T and $\widehat{T}$ resp. (=so called semistandard groups), a bijection which can be made explicit and which moreover is equivariant with respect to the $\mu(\Gamma)$-action on both sides. So it restricts to a bijection between $\mu(\Gamma)$-stable objects on both sides. But then we can form ${}^LP = \widehat{P} \rtimes_\mu W_F$ and ${}^LM = \widehat{M} \rtimes_\mu W_F$ and identify the $\mu(\Gamma)$-stable objects with the objects ${}^LP$, ${}^LM$ of the ${}^LT$-apartment of ${}^LG$.
Finally we may choose the basic data for the F-structure and $\overline{F}$-structure in a compatible way:

$$P_0 \supseteq B \supset T \supseteq A_0, \qquad P_0 = M_0 \cdot B,$$

and then the T-objects in G which are defined over F are precisely those $\mu(\Gamma)$-stable objects which contain M_0, and they transfer to those objects ${}^LP$, ${}^LM$ in the ${}^LT$-apartment which contain ${}^LM_0$. These objects are called the **relevant** parabolic and Levi-subgroups in ${}^LG$.
If (G, B, T) is quasisplit then we get $M_0 = T$ and all $\mu(\Gamma)$-stable objects are automatically defined over F, equivalently all objects in the ${}^LT$-apartment of ${}^LG$ are relevant.

Next we recall $\Phi(G)$ **the set of L-parameters for** $G = G(F)$. An L-homomorphism for G is a homomorphic map

$$\phi : SL_2(\mathbb{C}) \times W_F \to {}^LG = \widehat{G} \rtimes W_F.$$

The left side being a direct product we may give $\phi = (\phi_1, \phi_2)$ as a pair of two maps where the images must commute. It is required that ϕ_1 is a complex analytic map, so we come down to $\phi_1 : SL_2(\mathbb{C}) \to \widehat{G}$ because the image of connected must be connected again. On the other hand
$\phi_2 : W_F \to {}^LG$ should be of type $\phi_2(\gamma) = (\varphi_2(\gamma),\ \gamma) \in \widehat{G} \rtimes W_F$, where $\varphi_2 \in Z^1(W_F, \widehat{G})$ is a 1-cocycle.
Additional requirements: All $\phi_2(\gamma) \in {}^LG$ should be semisimple elements.
If (G, B, T) is not quasisplit, then ϕ should be **relevant**, this means if $Im(\phi) \subset L({}^LG)$ is contained in some Levi subgroup of ${}^LG$ then this group should be conjugate to a relevant Levi-group in the above sense. Roughly speaking, $Im(\phi)$ should not be too small.
For $x \in \widehat{G}$ and $\phi : SL_2(\mathbb{C}) \times W_F \to {}^LG$ a conjugate map $x\phi x^{-1}$ is well defined, and an L-parameter $[\phi]$ is by definition a conjugacy class of L-homomorphisms; the set of all L-parameters is denoted $\Phi(G)$.
Independent from the notion of tempered representation we recall:
Definition: An L-parameter $[\phi]$, $\phi = (\phi_1, \phi_2)$ is said to be **tempered** if the image of the cocycle $\varphi_2 : W_F \to \widehat{G}$ has compact closure.
Note here that the inertia subgroup $I \subset W_F$ is compact, and therefore if ϕ is not tempered this must be due to some $\varphi_2(\gamma)$ for $\gamma \in W_F - I$, in particular it is enough to check the Frobenius-lifts.

It is of course part of the Langlands conjectures that an L-packet Π_ϕ will consist of tempered representations if the parameter $[\phi]$ is tempered.
Example: If $G = GL_n(F)$ then $\widehat{G} = GL_n(\mathbb{C})$ and the Galois action on $\widehat{G}$ is trivial. So we have ${}^LG = GL_n(\mathbb{C}) \times W_F$ a direct product, and we may replace $\phi = (\phi_1, \phi_2)$ by the homomorphism

$$\phi = (\phi_1, \varphi_2) : SL_2(\mathbb{C}) \times W_F \to GL_n(\mathbb{C})$$

and then $[\phi]$ is nothing else then an equivalence class of n-dimensional representations of $SL_2(\mathbb{C}) \times W_F$. Then ϕ rewrites as

$$\phi = \bigoplus_i r_{n_i} \otimes \rho_i,$$

where r_{n_i} is the (up to equivalence unique) irreducible analytic representation of $SL_2(\mathbb{C})$ of dimension n_i and ρ_i is an irreducible representation of W_F by means of semisimple elements, and $\sum_i n_i \cdot dim(\rho_i) = n$. Then we have $\varphi_2 = \phi|_{W_F} = \bigoplus_i n_i\rho_i$, and tempered means that all the representations ρ_i are unitary. The Steinberg representation St_n has simply $\phi = r_n \otimes \mathbf{1}$, hence $\varphi_2 = n \cdot \mathbf{1}$ as its L-parameter.

Now an **L-parameter standard triple for** G should consist of $(P, [{}^t\phi], \nu)$ where P is a standard F-parabolic subgroup, $[{}^t\phi] \in \Phi(M_P)$ is a tempered L-parameter for the standard Levi subgroup M_P and $\nu \in \mathbb{R} \otimes X^*(M_P) = \mathfrak{a}^*_{M_P}$ is regular with respect to P.
Our Main Result is now:
Langlands classification for L-parameters: *There is a well defined bijection*

$$\Phi(G) \longleftrightarrow \{(P, [{}^t\phi], \nu)\}$$

between L-parameters of G and the set of L-parameter standard triples.

3. How to organize the Langlands classification for L-parameters

For representations σ of a Levi group $M = M(F)$ and $\nu \in \mathbb{R} \otimes X^*(M)_F$ we had
$\nu \mapsto \chi_\nu$=unramified positive real valued character of M,
$\sigma \mapsto \sigma \otimes \chi_\nu$=twist operation.

Now we replace σ by an L-parameter $[\phi] \in \Phi(M)$ and we want to define the twist of $[\phi]$ by ν.

Step 1: (see [A],p.201) Let $Z({}^LM)^0$ be the connected center of the L-group ${}^LM$. Then $[\phi] \in \Phi(M)$ can be twisted by an element $z \in Z({}^LM)^0$ as follows:

$$\phi = (\phi_1, \phi_2) \mapsto \phi_z := (\phi_1, \phi_2')$$

such that $\gamma \in W_F \mapsto \phi_2'(\gamma) := \phi_2(\gamma) \cdot z^{d(\gamma)} \in {}^LM$ where $d : W_F \to \mathbb{Z}$ is the Frobenius exponent.

Step 2: As part of the construction of $(G, B, T) \mapsto (\widehat{G}, \widehat{B}, \widehat{T})$ we have:

$$\widehat{T} \cong \mathbb{C}^\times \otimes X_*(\widehat{T}) = \mathbb{C}^\times \otimes X^*(T).$$

If $M \supseteq T$ is an F-Levi subgroup of G, then this induces a description of the subtorus $Z({}^LM)^0 \subset \widehat{T}$ as follows:

$$Z({}^LM)^0 \cong \mathbb{C}^\times \otimes X_*(Z({}^LM)^0) \cong \mathbb{C}^\times \otimes X^*(M)_F.$$

We restrict to hyperbolic elements in $Z({}^LM)^0$ which means all eigenvalues should be positive real numbers. Then we obtain:
(1)

$$Z({}^LM)^0_{hyp} \cong (\mathbb{R}_+)^\times \otimes X^*(M)_F \cong \mathbb{R} \otimes X^*(M)_F = \mathfrak{a}_M^*$$

where from right to left we have an exponential map:

$$\nu = s \otimes \chi \mapsto q^{s \otimes \chi} \mapsto z(\nu) := \chi^\vee(q^s),$$

where $\chi^\vee \in X_*(Z({}^LM)^0)$ is the cocharacter corresponding to $\chi \in X^*(M)_F$.

Step 3: Now the twist of $[\phi] \in \Phi(M)$ by $\nu \in \mathfrak{a}_M^*$ is explained as the twist $\phi_{z(\nu)}$ in the sense of step 1.

This gives us a well defined map from L-parameter standard triples to L-parameters
(2)

$$(P, [{}^t\phi]_M, \nu) \mapsto [\phi] := [{}^t\phi_{z(\nu)}]_G \in \Phi(G).$$

The difficult part is the construction of the converse map assigning an L-parameter standard triple to a given L-parameter. So we begin from an L-homomorphism $\phi : SL_2(\mathbb{C}) \times W_F \to {}^L G$. By definition the elements $\phi_2(\gamma) = (\varphi_2(\gamma), \gamma) \in {}^L G$ are always semisimple. Therefore they have a well defined polar decomposition:

$$\phi_2(\gamma) = \phi_2(\gamma)_h \cdot \phi_2(\gamma)_e$$

where the hyperbolic factor is in $\widehat{G}$, and the elliptic factor has the form $\phi_2(\gamma)_e = (\phi_2(\gamma)', \gamma)$.

Proposition 1, see [H], 5.1. *Let ϕ be as above, $\gamma_1 \in W_F$ a Frobenius lift, $s := \phi_2(\gamma_1)$ and $s_h \in \widehat{G}$ the hyperbolic part of s. Then:*
(i) *$s_h \in Z(C_{\widehat{G}}(Im(\phi))^0$ is in the central torus of the centralizer $C_{\widehat{G}}(Im(\phi))$.*
(ii) *s_h does not depend on the choice of the Frobenius lift γ_1.*
We may write now $z(\phi) := \phi_2(\gamma_1)_h$ because the result does not depend on the choice of γ_1. So with ϕ we have assigned a well defined semisimple hyperbolic element $z(\phi) \in \widehat{G}$.

Given $\phi \mapsto z(\phi)$ we ask now for Levi subgroups $L({}^L G)$ in ${}^L G$ such that
(3)

$$Im(\phi) \subset L({}^L G), \qquad z(\phi) \in Z(L({}^L G))^0.$$

Proposition 2: *For a given L-homomorphism ϕ there is precisely one* **maximal** *Levi subgroup $L({}^L G)_\phi$ subject to the conditions* **(3)***, namely*

$$L({}^L G)_\phi = C_{{}^L G}(z(\phi)),$$

the centralizer of $z(\phi)$ in ${}^L G$.
Remark: Usually the centralizer of a semisimple element need not be a Levi subgroup but here our element $z(\phi)$ is semisimple hyperbolic.
Since $[\phi] = \{x\phi x^{-1} \mid x \in \widehat{G}\}$ and since the map $\phi \mapsto (z(\phi),\ L({}^L G)_\phi)$ is compatible with conjugation we obtain a conjugacy class of pairs which is assigned to an L-parameter $[\phi]$.

Our problem will be settled now by the following
Proposition 3: *Given $[\phi] \in \Phi(G)$ the conjugacy class contains a well defined subset of representatives ϕ such that:*
(i) *$L({}^L G)_\phi = {}^L M$ is the L-group of a standard F-Levi subgroup $M \subseteq G$.*
(ii) *$z(\phi) = z(\nu) \in Z({}^L M)^0$ corresponds via* **(1)** *to an element $\nu \in \mathfrak{a}_M^*$ which is regular with respect to the standard F-parabolic group $P = M \cdot P_0$.*

Since $Im(\phi) \subset L({}^L G)_\phi = {}^L M$, we may consider ${}^t\phi := \phi_{z(\phi)^{-1}}$ as a tempered L-parameter of M, (note here that non-temperedness is caused by the hyperbolic part of $\phi_2(\gamma_1)$ which we have twisted away now) and
$(P, [{}^t\phi]_M, \nu)$ is the L-parameter standard triple which is assigned to $[\phi]$.

4.Example:

Let $G_n := GL_n(F)$ be the linear group, hence ${}^L G_n = GL_n(\mathbb{C}) \times W_F$. We consider

$\mathbf{1}_n \in Irr(G_n)$ the trivial representation. Moreover we consider $B_n,\ T_n$ the subgroups of upper triangular and of diagonal matrices resp. and in $X^*(T_n) = Hom(T_n, GL_1)$ we take e_i as the projection on the i-th coordinate, and we put $\mathfrak{a}_n^* = \mathbb{R} \otimes X^*(T_n)$. Then:
(i) The standard triple $(B_n,\ \mathbf{1}_{T_n},\ \nu = \frac{n-1}{2}e_1 + \cdots + \frac{1-n}{2}e_n)$ represents the Langlands classification of $\mathbf{1}_n$.
(ii) The L-parameter $\phi = (\phi_1, \varphi_2) = \phi(\mathbf{1}_n)$ is (up to equivalence) the representation $\phi : SL_2(\mathbb{C}) \times W_F \to GL_n(\mathbb{C})$ which is trivial on $SL_2(\mathbb{C})$ and sends $\gamma \in W_F$ to the diagonal matrix $\varphi_2(\gamma) = diag(q^{(n-1)/2}, ..., q^{(1-n)/2})^{d(\gamma)}$.
(iii) Let $\phi^0_{T_n} \in \Phi(T_n)$ be the trivial L-parameter (i.e. $\phi^0_{T_n} : SL_2(\mathbb{C}) \times W_F \to \widehat{T}_n$ is the trivial map). Then the L-parameter standard triple $\big(B_n,\ \phi^0_{T_n},\ \nu = \frac{n-1}{2}e_1 + \cdots \frac{1-n}{2}e_n\big)$ represents the Langlands classification of the L-parameter $\phi(\mathbf{1}_n) \in \Phi(G_n)$.
Proof: The triple written down in (i) is a standard triple, (we omit here to check that the $\nu \in \mathfrak{a}_n^*$ is indeed regular with respect to B_n) and

$$\chi_\nu = |.|_F^{(n-1)/2} \otimes \cdots \otimes |.|_F^{(1-n)/2} \in X_{ur}(T_n)$$

is the corresponding unramified character. Therefore (i) says that

$$\mathbf{1}_n = j(i_{G_n,B_n}(\chi_\nu))$$

is the unique irreducible quotient of the normalized parabolic induction.
In terms of the Bernstein Zelevinsky classification we have $\mathbf{1}_n = L(|.|_F^{(1-n)/2}, ..., |.|_F^{(n-1)/2})$, where each character is its own segment. Therefore the L-parameter will be:

$$\phi(\mathbf{1}_n) = \phi(L(|.|_F^{(1-n)/2}, ..., |.|_F^{(n-1)/2})) = \bigoplus_i \phi(L(|.|_F^i)) = \bigoplus \omega_i,$$

where $\omega_i : W_F \to \mathbb{C}^\times$ corresponds to $|.|_F^i : F^\times \to \mathbb{C}^\times$ via class field theory (renormalized in such a way that inverses of Frobenius lifts correspond to prime elements), i.e. $\omega_i(\gamma) = q^{id(\gamma)}$.
Finally if we take $\phi = \phi(\mathbf{1}_n)$ then up to equivalence (interchanging the summands) we may assume:

$$z(\phi) = \phi(\gamma_1)_h = diag(q^{(n-1)/2}, ..., q^{(1-n)/2}) \in Z(\,{}^LT_n)^0 = \widehat{T}_n,$$

and the centralizer is $C_{{}^LG_n}(z(\phi)) = \widehat{T}_n \times W_F = {}^LT_n$. Then $\phi_{z(\phi)^{-1}} = \phi^0_{T_n}$ will be the trivial L-parameter of T_n, and $z(\phi) = z(\nu)$ for $\nu = \frac{n-1}{2}e_1 + \cdots + \frac{1-n}{2}e_n \in \mathfrak{a}_n^*$. **qed.**

In general, with $(G,\ A_0,\ M_0,\ P_0)$ as above, the trivial representation $\mathbf{1}_G$ is assigned to the standard triple $(P_0, \mathbf{1}_{M_0},\ log(\delta_{P_0}^{1/2}))$, where $\delta_{P_0} \in X_{ur}(M_0)$ denotes the modular character of P_0. This character is actually a positive real valued unramified character of M_0 which arises from the difference between left and right Haar measures on P_0, and the notation $\nu := log(\delta_{P_0}^{1/2}) \in \mathfrak{a}_{M_0}^*$ stands for the relation $\chi_\nu = \delta_{P_0}^{1/2}$.
Since M_0 is compact modulo center (it has no F-parabolic subgroups), the trivial representation $\mathbf{1}_{M_0}$ is square integrable, in particular it is tempered. What is now the L-parameter

$\phi(\mathbf{1}_{M_0}) \in \Phi(M_0)$? The problem is that $\phi(\mathbf{1}_{M_0})$ should be relevant, the image should not be to small. We use here the fact that each reductive F-group is the inner form of a quasisplit group, and groups which are inner forms of each other have the same L-group. So we have ${}^L M_0' = {}^L M_0$ if M_0' denotes the quasisplit inner form of M_0. Now one expects that

$$\phi(\mathbf{1}_{M_0}) : SL_2(\mathbb{C}) \times W_F \to {}^L M_0 = {}^L M_0'$$

identifies with the L-parameter of the Steinberg representation of M_0'. Because the Steinberg representation is square integrable, its L-parameter ϕ is maximal in the sense that $Im(\phi)$ does not fit into any proper Levi-subgroup of ${}^L M_0'$.
For instance if $M_0 = D^\times$ is a divison algebra of index d then the trivial representation of $D^\times$ corresponds to the Steinberg representation of $M_0' = G_d$ and the corresponding L-parameter is the irreducible representation r_d of $SL_2(\mathbb{C})$ of dimension d.

References

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For details and more references see: arXiv:1407.6494 [math.RT]