
Zbl pre05190930

Grabitz, Martin

Simple characters for principal orders in $M_m(D)$. (English)

J. Number Theory 126, No. 1, 1-51 (2007).
http://dx.doi.org/10.1016/j.jnt.2006.06.008
http://www.sciencedirect.com/science/journal/0022314X

The paper, which became the author's mathematical testament, gives an independent development of the subject of strata and simple characters for local central simple algebras A. More precisely the center of A is a p-adic field F. The final aim (not in this paper) is to construct irreducible representations of the p-adic group $A^\times$ via open compact subgroups $\mathcal{K}$. For discrete series representations one expects that we may restrict to groups $\mathcal{K} = \mathfrak{A}^\times$, where $\mathfrak{A} \subset A$ is a principal order (a hereditary o_F-order in A such that its Jacobson radical $\mathfrak{P}$ is a principal ideal in $\mathfrak{A}$). This is why the author's approach dealing only with principal orders $\mathfrak{A}$ is still sufficiently general.

There are three main results of the paper: (i) the construction of simple characters (ii) the matching of simple characters between different simple algebras (iii) the intertwining implies conjugacy property of simple characters. In the case of (i) and (ii) the more recent work of V. Secherre, partly in collaboration with S. Stevens goes further, also including the construction of (maximal) simple types [see *V. Secherre* and *S. Stevens*, "Représentations lisses de $GL_m(D)$ IV: représentations supercuspidale", J. Inst. Math. Jussieu 7, No. 3, 527–574 (2008; Zbl 1140.22014), and the references therein].

However we mention that also the author was partly successful in constructing simple types [*M. Grabitz*, Simple characters for principal orders, part II, preprint MPIM2003-56, Max-Planck-Institut, Bonn (2003)] and that prior to Grabitz's work, (iii) had been known only in the split case, in the book of *C. J. Bushnell* and *P. C. Kutzko* [The admissible dual of $GL(N)$ via compact open subgroups, Annals of Mathematics Studies 129 (1993; Zbl 0787.22016)]. The author follows in many respects the approach of that treatise [BK] which will be the main reference. By Wedderburn's theorem any central simple algebra can be identified with a matrix algebra over a central division algebra, hence $A \cong M_m(D)$ and the split case is the case when $D = F$.

To put the paper into perspective one needs to refer to the program anticipated by Bushnell and Kutzko, to construct types for reductive p-adic groups. The category $\mathcal{M}(G)$ of smooth representations of such groups decomposes into a direct product of subcategories – the Bernstein blocs. The aim is to find for each Bernstein bloc $\mathcal{M}$ a pair $(\mathcal{K}, \rho)$ consisting of an open compact subgroup $\mathcal{K} \subset G$ and an irreducible representation ρ of $\mathcal{K}$ such that $\mathcal{M}$ can be identified with the category of representations which are generated by their ρ-isotypic component. Then $(\mathcal{K}, \rho)$ is called a type for $\mathcal{M}$. In general it is not known whether each Bernstein bloc $\mathcal{M}$ of $\mathcal{M}(G)$ has a type. On the other hand if a component has a type, the type is not unique.

The construction of a complete set of types for $G = GL_N(F)$ is one of the main achievements in the work of Bushnell and Kutzko. One has to distinguish simple types which refer to blocs $\mathcal{M}$ which contain the discrete series representations of G and semisimple types which concern the other blocs. In his thesis (July 2000) the author made an important step forward toward constructing simple types for the groups $G = A^\times$. The paper under review summarizes his results on simple characters including the more recent results on the intertwining implies conjugacy property of simple characters. These results are preliminary to the construction of simple types.

The simple types are expected as representations of compact groups $\mathcal{K} = \mathfrak{A}^\times$ the unit groups of principal orders in the matrix algebra $A = M_m(D)$ (which can be considered as the Lie algebra of G). Since $\mathfrak{A}^\times$ is provided with a natural filtration $\mathfrak{A}^\times \supset U^1_{\mathfrak{A}} \supset U^2_{\mathfrak{A}} \supset \cdots$ of normal subgroups, the higher congruence or principal unit subgroups $U^i_{\mathfrak{A}} = 1 + \mathfrak{P}^i$, each irreducible representation ρ of $\mathfrak{A}^\times$ is naturally induced from a subgroup $J = J_\rho \subset \mathfrak{A}^\times$, such that the inducing representation ρ_J is isotypic on all subgroups $J^i = J \cap U^i_{\mathfrak{A}}$ of the intersected filtration. Then together with $(\mathcal{K}, \rho)$ also (J, ρ_J) is a simple type for the same Bernstein bloc $\mathcal{M}$. By very general reasons J has normal subgroups $J^1 \supset H^1$ such that $J/H^1 = J^1/H^1 \rtimes H/H^1$ is the semidirect product of the abelian quotient J^1/H^1 with a distinguished complement H/H^1.

As suggested by results of L. Corwin, H. Reimann and J.-K. Yu one may expect that the simple type ρ_J is determined by a *simple character* θ of H in such a way that

(i) the restriction of ρ_J to J^1 is the unique irreducible representation η of J^1 which on H^1 is a multiple of θ,

(ii) the restriction of ρ to H is $\theta \otimes W$, where W is a Weil representation of the quotient group H/H^1. Via θ one has a symplectic form on the $\mathbb{F}_p$-vector space J^1/H^1 (see 5.6.), and the action of H/H^1 by conjugation fixes this symplectic form. In this context a Weil representation of H/H^1 is well defined if we assume p-adic for $p \neq 2$.

The approach of [BK] and V. Secherre resp. to simple types is different and covers also the case $p = 2$. There it is important that Secherre works with all hereditary orders. On the other hand, the author's approach to simple types (see Grabitz, loc.cit.) comes close to what has been sketched here. He improves the results of H. Reimann where the method is to get first an extension ρ_{J_p} on a p-Sylow subgroup $J_p \subset J$ (note $J_p/H^1 = J^1/H^1 \rtimes H_p/H^1$). The extension from J_p to J is then well determined because $\dim(\rho_J) = \sqrt{(J^1 : H^1)}$ is a power of p.

The paper under review deals with the simple characters. After preparations in sections 1–4, the first topic is in section 5 the construction of a set $\mathcal{C}(\beta, \mathfrak{A})$ of simple characters to any simple stratum $[\mathfrak{A}, n, 0, \beta]$. A stratum in A is ((1.8) similar as in [BK]) a quadruple $[\mathfrak{A}, n, \nu, b]$ where now $\mathfrak{A}$ is always a principal o_F-order in A, $\nu \leq n$ are integers and $b \in \mathfrak{P}^{-n} \supseteq \mathfrak{P}^{-\nu}$. One may think of a stratum as a representative for a residue class $[b] \in \mathfrak{P}^{-n}/\mathfrak{P}^{-\nu}$. Equivalence $[\mathfrak{A}, n, \nu, b_1] \sim [\mathfrak{A}, n, \nu, b_2]$ means $b_1 - b_2 \in \mathfrak{P}^{-\nu}$, and for convenience we introduce here the partial order for equivalence classes:

$$[\mathfrak{A}, n, \nu_1, b_1] \lesssim [\mathfrak{A}, n, \nu_2, b_2] \quad \text{if } \nu_1 \geq \nu_2 \text{ and } b_1 - b_2 \in \mathfrak{P}^{-\nu_1}$$

in other words, if we have a projection backwards: $\mathfrak{P}^{-n}/\mathfrak{P}^{-\nu_2} \to \mathfrak{P}^{-n}/\mathfrak{P}^{-\nu_1}$ which takes $[b_2]$ into $[b_1]$. The stratum $[\mathfrak{A}, n, \nu, \beta]$ is said to be pure (3.1), if $\nu_{\mathfrak{P}}(\beta) = -n$, and if β generates a subfield of A such that $F[\beta]^\times$ normalizes $\mathfrak{A}$ and if moreover there exists $E|F[\beta]$ a maximal subfield in A which also normalizes $\mathfrak{A}$.

A pure stratum $[\mathfrak{A}, n, \nu, \beta]$ has the degree $[F[\beta] : F]$. It is called simple if it has minimal degree among all equivalent pure strata $[\mathfrak{A}, n, \nu, b] \sim [\mathfrak{A}, n, \nu, \beta]$. A pure stratum

$[\mathfrak{A}, n, \nu, \beta]$ is simple as soon as $\nu < -k_0(\beta, \mathfrak{A})$ where $k_0(\beta, \mathfrak{A})$ is a well defined critical exponent. In particular simplicity is preserved if we make ν smaller.
An approximation sequence (3.9) for a simple stratum $[\mathfrak{A}, n, 0, \beta]$ is a sequence of simple strata

$$[\mathfrak{A}, n, q_{\mu-1}, \gamma_{\mu-1}] \lesssim \cdots \lesssim [\mathfrak{A}, n, q_1, \gamma_1] \lesssim [\mathfrak{A}, n, 0, \beta]$$

such that going from right to left $[\mathfrak{A}, n, q_i - 1, \gamma_{i-1}]$ is still simple but $[\mathfrak{A}, n, q_i, \gamma_{i-1}]$ is not and is replaced by the equivalent simple stratum $[\mathfrak{A}, n, q_i, \gamma_i]$ of smaller degree. The first stratum $[\mathfrak{A}, n, q_{\mu-1}, \gamma_{\mu-1}]$ is minimal, which means it is of degree 1, or the critical exponent is $-n$. An approximation sequence is minimal in the sense that a representative γ_i is changed only if it is absolutely necessary, and then from right to left the degree of the stratum becomes smaller, actually divides by a factor. Longer sequences where the representatives are changed more often are called weak approximation sequences.
In section 4 two subgroups $H^1(\beta, \mathfrak{A}) \subseteq J^1(\beta, \mathfrak{A})$ of $U^1_{\mathfrak{A}}$ are assigned with a simple stratum $[\mathfrak{A}, n, 0, \beta]$. (These groups play the role of J^1, H^1 in the above discussion.) The construction is similar as in [BK](3.1) but the method of orbit filters makes it clear that the assignment is natural. In particular it only depends on the equivalence class of the stratum and has several nice properties, for instance

$$U_{\mathfrak{A}}^{[q/2]+1} \cap C_A(\gamma) \subseteq H^{[q/2]+1}(\gamma, A) = H^{[q/2]+1}(\beta, A) \quad \text{if } [\mathfrak{A}, n, q, \gamma] \lesssim [\mathfrak{A}, n, 0, \beta],$$

where the notation $H^i(\beta, \mathfrak{A}) = U^i_{\mathfrak{A}} \cap H^1(\beta, \mathfrak{A})$ is used, and $C_A(\gamma)$ denotes the centralizer of γ.
The construction of the set $\mathcal{C}(\beta, \mathfrak{A})$ of simple characters to a simple stratum $[\mathfrak{A}, n, 0, \beta]$ is now similar as in §3 of [BK] but the details are more subtle, and results of P.Broussous on the intertwining of strata are applied. One uses weak approximation sequences and the relation

$$(*) \qquad H^1(\beta, \mathfrak{A}) = (U^1_{\mathfrak{A}} \cap C_A(\beta)) \cdot H^{[q/2]+1}(\gamma, \mathfrak{A})$$

if $[\mathfrak{A}, n, q, \gamma] \lesssim [\mathfrak{A}, n, 0, \beta]$ is close enough. By induction the character set $\mathcal{C}(\gamma, \mathfrak{A})$ has been defined already and one wants $\theta_\beta \in \mathcal{C}(\beta, \mathfrak{A})$ such that

$$(\theta_\beta \theta_\gamma^{-1})(1 + x) = \psi_F \circ Trd_{A|F}((\beta - \gamma)x)$$

for $1 + x \in H^{[q/2]+1}(\beta, \mathfrak{A}) = H^{[q/2]+1}(\gamma, \mathfrak{A})$, and where ψ_F is an additive character of the base field. The author shows (Main Lemma 5.2) that the character $\theta_\gamma \cdot \psi^\times_{\beta-\gamma}$ on the second factor of the decomposition $(*)$ is invariant under $\mathfrak{A}^\times \cap C_A(\beta)$, and can be extended to the first factor by means of a character $\lambda_\beta \circ Nrd_{C_A(\beta)|F[\beta]}$ which comes from a character λ_β on $F[\beta]^\times$. The inductive proof depends on computing at the same time the intertwining of θ_β restricted to the various subgroups $H^\nu(\beta, \mathfrak{A})$. ((5.8) and (5.9)). The final results are:
(5.14) Constructing θ_β by means of a fixed weak approximation sequence $[\mathfrak{A}, n, q_i, \gamma_i] \lesssim [\mathfrak{A}, n, 0, \beta]$ one finds that θ_β restricted to $U_{\mathfrak{A}}^{[q_i/2]+1} \cap C_A(\gamma_i)$ has the form $(\lambda_{\gamma_i} \circ Nrd) \cdot \psi^\times_{\beta-\gamma_i}$ where λ_{γ_i} is a uniquely determined character of $U_{\mathfrak{A}}^{[q_i/2]+1} \cap F[\gamma_i]^\times$. And conversely θ_β is determined by a compatible sequence of characters λ_{γ_i}. Such a system $\{(\gamma_i, \lambda_{\gamma_i})\}_{i=0}^{\mu-1}$ is called a defining sequence (5.4).

(5.11) The definition of the set $\mathcal{C}(\beta, \mathfrak{A})$ of characters of $H^1(\beta, \mathfrak{A})$ depends only on the equivalence class of the stratum $[\mathfrak{A}, n, 0, \beta]$ and does not depend on the choice of the weak approximation sequence.

The second achievement of the paper is in sections 6, 7 the matching of simple characters between various algebras via generalized transfer maps. It starts from the notion [introduced by *C. Bushnell* and *G. Henniart*, "Local tame lifting for $GL(N)$, I: Simple characters", Publ. Math., Inst. Hautes Étud. Sci. 83, 105–233 (1996; Zbl 0878.11042)] of a simple pair $[0, \beta]$ for β in a fixed algebraic closure $\overline{F}$ with $E := F[\beta]$ and $n_0 = \nu_E(\beta) < 0$. There is a natural split algebra $A(E) := \mathrm{End}_F(E)$ where we can embed E and a natural principal order $\mathfrak{A}(E) = \bigcap_{i \in \mathbb{Z}} \mathrm{End}_{o_F}(\mathfrak{p}_E^i)$ such that the embedding is $\mathfrak{A}(E)$-pure, and simple pair means that the natural stratum $[\mathfrak{A}(E), n_0, 0, \beta]$ in $A(E)$ is simple. Now if $A|F$ and $\mathfrak{A} \subset A$ are any central simple algebra and any principal order such that we can find an embedding $\iota(\beta) \in A$ which is $\mathfrak{A}$-pure, then the corresponding stratum $[\mathfrak{A}, n, 0, \beta]$ is always simple.

The main result (Theorem 7.2.) says, that under these assumptions we get always a natural bijection
$$\tau_\iota : \mathcal{C}(\beta, \mathfrak{A}(E)) \to \mathcal{C}(\iota(\beta), \mathfrak{A})$$
between the simple characters of the groups $H^1(\beta, \mathfrak{A}(E))$ and $H^1(\iota(\beta), \mathfrak{A})$ resp. which is characterized by the property that a defining sequence for the character $\theta_\beta \in \mathcal{C}(\beta, \mathfrak{A}(E))$ can be transported by a series of embedding maps into a defining sequence for the character $\tau(\theta_\beta)$.

The proof needs the notion of special approximation sequences (see 3.12). One begins from a K-special approximation for $[\mathfrak{A}(E), n_0, 0, \beta]$ where K is the inertia subfield of $E|F$, and shifts it to a $\iota(K)$-special approximation of $[\mathfrak{A}, n, 0, \iota(\beta)]$. These approximations can be intersected down to the centralizer of K and $\iota(K)$ respectively. Over the new base field all extension fields are totally ramified. This is precisely the situation section 6 refers to, where it is possible (similar as before Bushnell and Henniart, loc.cit. in the split case) to establish an unramified base change for simple characters with underlying totally ramified simple stratum, a base change which at the same time splits the algebra A. This gives a reduction to the split case where the maps τ had been introduced in [BK], 3.6.

Finally the notion of a ps-character introduced by Bushnell and Henniart can be extended as follows:

To a simple pair $[0, \beta]$ and a simple character $\theta_\beta \in \mathcal{C}(\beta, \mathfrak{A}(E))$ one has a character valued function $(A, \iota, \mathfrak{A}) \mapsto \theta(A, \iota, \mathfrak{A}) := \tau_\iota(\theta_\beta) \in \mathcal{C}(\iota(\beta), A)$ which is defined on each triple $(A, \iota, \mathfrak{A})$ fulfilling the assumptions of Theorem 7.2. Fixing A and $\iota(\beta) \in A$ it might be possible that the assumptions are fulfilled for several different orders $\mathfrak{A}$. In section 8 it is proved that $\theta(A, \iota, \mathfrak{A}_1)$, $\theta(A, \iota, \mathfrak{A}_2)$ agree on $H^1(\iota(\beta), \mathfrak{A}_1) \cap H^1(\iota(\beta), \mathfrak{A}_2)$ if the intersection $\mathfrak{A}_1 \cap \mathfrak{A}_2 \cap C_A(\iota(\beta))$ is large enough to contain a minimal order of the centralizer $C_A(\iota(\beta))$. (8.11) This result plays an important role in the author's construction of simple types (see Grabitz, loc.cit. sections 3 and 4).

The last two sections 9 and 10 present the third important result, which has been obtained by the author more recently, namely the intertwining implies conjugacy property of simple characters.

In fact one wants to know this for the simple types (J, ρ_J) mentioned at the beginning (and for modifications of them which are obtained by level zero twists.) Each such type is expected to determine a Bernstein bloc $\mathcal{M}$ of the category $\mathcal{M}(A)$ in such a way that an irreducible representation Π of $A^\times$ contains ρ_J if and only if $\Pi \in \mathcal{M}$. Therefore the types (J_1, ρ_1), (J_2, ρ_2) must intertwine: there exists $x \in A^\times$ such that $\mathrm{Hom}_{x^{-1}J_1x\cap J_2}(\rho_1^x, \rho_2) \neq 0$, if they determine the same $\mathcal{M}$. One wants to know that in such a case the types are actually conjugated: $(J_2, \rho_2) = (x^{-1}J_1x, \rho_1^x)$. This would give an injection of the set of conjugacy classes from our selected list of types into blocs $\mathcal{M}$ which contain discrete series representations. In fact one expects that each $\mathcal{M}$ contains up to unramified twist only one discrete series representation, such that we also have an injection of conjugacy classes of types into the set of discrete series representations modulo unramified twist. Moreover under this injection the (normalized) conductor of the type agrees with the conductor of the corresponding discrete series representation.

But from the Abstract Matching Theorem of Deligne, Kazhdan, Vigneras, Badulescu it is known that we have a conductor-preserving bijection between discrete series representations for algebras A_1, A_2 of the same size which is also compatible with character twist, and moreover there are only finitely many unramified twist classes of discrete series representations if we fix the conductor. This suggests that it must be possible to show that the injection from classes of types to twist classes of discrete series representations is surjective for all A as soon as we know this for at least one A of each size , for instance the split A. It was one of the author's projects to confirm this. In Grabitz, loc.cit. section 12 he got a partial result in that direction.

As a preliminary result the author proves intertwining implies conjugacy for simple characters, a generalization of [BK](3.5.11) where the split case had been dealt with. For technical reasons one considers also the character sets $\mathcal{C}(\beta, \nu, \mathfrak{A})$ for $0 \leq \nu < n = -\nu_{\mathfrak{A}}(\beta)$ which arise by restricting the simple characters $\theta \in \mathcal{C}(\beta, \mathfrak{A})$ to the subgroup $H^{\nu+1}(\beta, \mathfrak{A})$. (5.11) includes the fact that $\mathcal{C}(\beta, \nu, \mathfrak{A})$ depends only on the equivalence class of the stratum $[\mathfrak{A}, n, \nu, \beta]$. In preparation for the main result, in section 9 results on rigidity and refinement of simple characters are proved which generalize similar results of [BK](3.5). If two character sets $\mathcal{C}(\beta_i, \nu, \mathfrak{A})$, $i = 1, 2$ intersect then they actually agree (9.9). Moreover (9.10) if $\mathcal{C}(\beta_1, \nu, \mathfrak{A}) = \mathcal{C}(\beta_2, \nu, \mathfrak{A})$ for some $\nu \geq 1$ ($\nu > 1$ is a misprint) then $\mathcal{C}(\beta_1', \nu - 1, \mathfrak{A}) = \mathcal{C}(\beta_2, \nu - 1, \mathfrak{A})$ for some $\beta_1' \in \beta_1 + \mathfrak{P}^{-\nu}$.

Finally (9.13) if $[\mathfrak{A}, n, q, \gamma] \lesssim [\mathfrak{A}, n, q-1, \beta]$ are simple strata of different degree and if $\theta \in \mathcal{C}(\beta, q-1, \mathfrak{A})$, $\phi \in \mathcal{C}(\gamma, q-1, \mathfrak{A})$ are characters which agree on $H^{q+1}(\beta, \mathfrak{A}) = H^{q+1}(\gamma, \mathfrak{A})$, then θ and ϕ do **not** intertwine in $A^\times$. (Since the intertwining property of two characters $\theta_i \in \mathcal{C}(\beta_i, \nu, \mathfrak{A})$, $i = 1, 2$ is preserved under restriction, in other words if we increase ν, one can use (9.13) to show that there is no intertwining of simple characters if the strata $[\mathfrak{A}, n, \nu, \beta_i]$ are of different degree. Also there can be no intertwining of simple characters if the numbers $n_i = -\nu_{\mathfrak{A}}(\beta_i)$ are different, because these numbers are related to the conductor.)

In section 10 the author proves his main result Theorem (10.3). We only quote the important Corollary (10.15) which generalizes [BK](3.5.11) to any central simple algebra (provided the order $\mathfrak{A}$ is principal): Let $[\mathfrak{A}, n, \nu, \beta_i]$ $i = 1, 2$ be two simple strata fulfilling the same notion of $\mathfrak{A}$-purity and assume that $\theta_i \in \mathcal{C}(\beta_i, \nu, \mathfrak{A})$, are simple characters which intertwine in $A^\times$, i.e. $\theta_{\beta_1}^x = \theta_{\beta_2}$ on $x^{-1}H^{\nu+1}(\beta_1, \mathfrak{A})x \cap H^{\nu+1}(\beta_2, \mathfrak{A})$ for some $x \in A^\times$. Then there exists an x normalizing $\mathfrak{A}$ which conjugates θ_1 into θ_2, and the character sets $\mathcal{C}(\beta_2, \nu, \mathfrak{A}) = \mathcal{C}(\beta_1^x, \nu, \mathfrak{A})$ agree because they have nontrivial intersection $\theta_2 = \theta_1^x$ (see above).

The proof is by induction on ν beginning from $\nu = n - 1$ and going down to $\nu = 0$. One uses a special approximation sequence $[\mathfrak{A}, n, q_i, \gamma_i] \lesssim [\mathfrak{A}, n, \nu, \beta]$. For $\nu \geq [\frac{n}{2}]$ simple strata (modulo equivalence) and simple characters can be naturally identified (in particular $\mathcal{C}(\beta, \nu, \mathfrak{A})$ is then a one-element-set) and the Theorem comes down to results of Broussous, Grabitz on intertwining of strata. In the induction step one has to distinguish the cases (A) that $[\mathfrak{A}, n, \nu, \beta_i]$ has an approximation sequence of the same length as needed for $[\mathfrak{A}, n, \nu + 1, \beta_i]$ or (B) it needs an approximation sequence which is longer. By the intertwining assumption and section 9 each of the two cases occurs simultaneously for both β_1 and β_2. Once (A) is proved the induction on ν can be combined with an induction on the length of approximation and then only (B) matters.

E. W. Zink

Keywords : representations of p-adic groups; principal orders in local central simple algebras; simple strata and simple characters; Bushnell Kutzko types
Classification :
∗22E50 Repres. of Lie and linear algebraic groups over local fields
11S45 Algebras and orders, and their zeta functions
11S37 Langlands-Weil conjectures, nonabelian class field theory
16H05 Orders and arithmetic, separable associative algebras